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THE PRICE OF ANARCHY

A SURVEY SUBMITTED FOR *TOPICS IN GAME THEORY* UNDER THE GUIDANCE OF DR TIOZZO

Contents

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Introduction

1.1 The tragedy of the commons – a brief history

In 1832, Reverend W.F. Lloyd delivered two lectures before the University of Oxford. Together, the lectures were entitled *Two Lectures on the Checks to Population*. The lectures mainly regarded Lloyd's concerns with population control and the concomitant risks. Nearing the end of his first lecture he made what would undoubtedly become his most notable remark

> Why are the cattle on a common so puny and stunted ? Why is the common itself so bare-worn, and cropped so differently from the adjoining inclosures ? No inequality, in respect of natural or acquired fertility, will account for the phenomenon. The difference depends on the difference of the way in which an increase of stock in the two cases affects the circumstances of the author of the increase. If a person puts more cattle into his own field, the amount of the subsistence which they consume is all deducted from that which was at the command, of his original stock ; and if, before, there was no more than a sufficiency of pasture, he reaps no benefit from the additional cattle, what is gained in one way being lost in another. But if he puts more cattle on a common, the food which they consume forms a deduction which is shared between all the cattle, as well that of others as his own, in proportion to their number, and only a small part of it is taken from his own cattle. In an inclosed pasture, there is a point of saturation, if I may so call it, (by which, I mean a barrier depending on considerations of interest,) beyond which no prudent man will add to his stock. In a common, also, there is in like manner a point of saturation. But the position of the point in the two cases is obviously different. Were a number of adjoining pastures, already fully stocked, to be at once thrown open, and converted into one vast common, the position of the point of saturation would immediately be changed. The stock would be increased, and would be made to press much more forcibly against the means of subsistence.[3]

In fact, Lloyd was trying to use the above metaphor to explain systems of inequality as it pertained to human labour.

Prescient as his lectures were, it took nearly 150 years before, in 1968, Lloyd's metaphor gained public relevance. In that year, the ecologist Garrett Hardin cited Lloyd in his article "The Tragedy of the Commons" published in *Science*. It was here that the phrase "Tragedy of the Commons" was coined and Hardin's article is often credited for the modern interpretation of the phrase. The relevant passage can be read below.

> As a rationale being, each herdsman seeks to maximize his gain. Explicitly or implicitly, more or less consciously, he asks, "what is the utility to me of adding one more animal to my herd?" This utility has a negative and a positive component.
>
> 1. The positive component is a function of the increment of one animal. Since the herdsman receives all the proceeds from the sale of the additional animal, the positive utility is nearly +1.
> 2. The negative component is a function of the additional overgrazing created by one more animal. Since, however, the effects of overgrazing are shared by all [...], the negative utility for any particular decision-making herdsman is only a fraction of -1.
>
> Adding together the component particular utilities, the rational herdsman concludes that the only sensible course for him to pursue is to add another animal to his herd. And another; and another... But this is the conclusion reached by each and every rational herdsman sharing a commons. Therein is the tragedy. Each man is locked into a system that compels him to increase his herd without limit – in a world that is limited. Ruin is the destination towards which all men rush, each pursuing his own best interest in a society that believes in the freedom of the commons. Freedom in a commons brings ruin to all.[2]

1.2 *Game-theory meets economics*

If one were to take the hypothetical game mentioned in Hardin's paper at face-value the corresponding payoff matrix would be [1]

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	$1, 1$	$1-\epsilon, 2-\epsilon$
	Defect	$2-\epsilon, 1-\epsilon$	$0, 0$

- If both players cooperate and have one cow each on the pasture, then both players receive a payoff of 1
- But player 1 can increase their payoff by putting another cow on the pasture
- Adding an extra cow would increase the payoff of player 1 to $1 + 1 - \epsilon = 2 - \epsilon$
  - Since putting another cow on the pasture also incurs a cost $-\epsilon$ to player 1 himself
- In the event that player 2 cooperates, they will receive a payoff of $1 - \epsilon$ (due to the incurred cost and no derived benefit from the cow placed on the pasture by player 1)
- By symmetry a similar payoff structure holds if player 2 defects and player 1 cooperates
- If both players defect then neither of them will receive a payoff since the pasture cannot support two additional cows

In light of the above payoff matrix it is clear that mutual defection is in fact *not* a Nash equilibrium. For these reasons and others Hardin's framework, although ambitious in its attempt to create a game-theoretic interpretation of the "Tragedy of the Commons", is often criticized for its ambiguity and internal inconsistencies. Regardless, the idea has still proven to be useful in other contexts such as traffic and network analysis as discussed here.

2

Setup

2.1 *General-sum games with more than two players*

Consider a game[4] with k players. Let S_i be a the set of pure strategies[1] of player i.

[1] Recall that a pure strategy $s_i \in S_i$ is a strategy that player i takes w.p.1

Definition 1 (Utility function). *A function*

$$u : S_1 \times \cdots S_l \to \mathbb{R}^k$$
$$s := (s_1, \ldots, s_k) \mapsto u(s) := u(s_1, \ldots, s_k) = \Big(u_1(s), \ldots, u_k(s)\Big)$$

is called a payoff/utility function .[2]

The pair $G = (S, u)$ defines a game.[3]

[2] Where $u_i(s)$ represents the payoff to player i

[3] Where $S = S_1 \times \cdots \times S_k$

2.2 *Price of anarchy*

Let $G = (S, u) = (S_1 \times \cdots S_k, u)$ be a game.

Definition 2 (Welfare). *A function*

$$W : S \to \mathbb{R}$$

is referred to as a welfare function .[4]

[4] W is defined in such a way as to be a measure of the "efficiency" of a given strategy $s \in S$. Given that "efficiency" is a subjective notion, W will depend on the game being played

W is conventionally defined in terms of the utility function u, for example,

$$W := \sum_i u_i$$

Definition 3 (Price of anarchy). *Let $E \subset S$ be the set of Nash equilibria of a game $G = (S, u)$. Let W be a corresponding welfare function. Then*

$$\max_{s \in S} W(s)$$

is defined as the optimal solution .[5] *Similarly,*

[5] "Optimum" here is synonymous with the *maximization of welfare*

$$\min_{s \in E} W(s)$$

is defined as the worst equilibrium .[6]

[6] The "worst equilibrium" is the Nash equilibrium with the lowest welfare rating

The ratio of these two values

$$\text{PoA} := \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)}$$

3
Selfish routing

3.1 Setup

Let $G = (V, E)$ be a directed graph. Let $s, t \in V$ be two given nodes. Let[1]

$$\mathscr{P}_{st} = \{\text{paths in } G \text{ from } s \text{ to } t\}$$

[1] $\mathscr{P}_{st}$ is a strategy profile

Let $P \in \mathscr{P}_{st}$ be a given path from s to t. Suppose that there are N total drivers. Let n_P be the number of drivers that take route P to travel from s to t. Let

$$f_P = \frac{n_P}{N}$$

be the proportion of drivers that take path P. Let

$$\mathbf{f} := (f_P)_{P \in \mathscr{P}_{st}}$$

3.2 All possible flows

Any indexed set $\mathbf{f} := (f_P)_{P \in \mathscr{P}_{st}}$ such that $f_P \geq 0$ and $\sum_{P \in \mathscr{P}_{st}} f_P = 1$ is a valid traffic flow. The set

$$\Delta(\mathscr{P}_{st}) := \left\{ \mathbf{f} : f_P \geq 0 \,\forall P \text{ and } \sum_{P \in \mathscr{P}_{st}} f_P = 1 \right\}$$

is the set of all possible traffic flows.

3.3 Latency

Let $e \in P \in \mathscr{P}_{st}$. Let x denote the fraction of drivers traveling along edge e. What is needed is a measure of the utility of traveling along a particular path P. The utility of traveling along a particular edge $e \in P$, as a function of the traffic x along that edge, is denoted as

$$\ell_e(x) \in \mathbb{R}$$

and referred to as a *latency function* .

Induced traffic

A given edge e may be shared by *multiple* paths $P \in \mathscr{P}_{st}$. The sum

$$F_e := \sum_{P|e \in P} f_P$$

is the traffic along the edge e. Then

$$\ell_e(F_e)$$

is the latency along the edge e when accounting for traffic from *all* paths that share edge e .

The latency of the *entire path* P is the sum of the latency of each of its edges

$$L_P(\mathbf{f}) = \sum_{e \in P} \ell_e(F_e)$$

Total latency

Since a fraction f_P of the traffic has latency $L_P(\mathbf{f})$, the total latency is

$$L(\mathbf{f}) = \sum_P f_P L_p(\mathbf{f})$$

This is the welfare function for this particular scenario as mentioned earlier.

3.4 *Nash equilibrium flow*

Definition 4 (Nash equilibrium flow)**.** *A flow* $\mathbf{f}$ *such that*

$$f_P > 0 \iff L_P(\mathbf{f}) = \min_{P' \in \mathscr{P}_{st}} L_{P'}(\mathbf{f})$$

is referred to as a Nash equilibrium flow .[2]

[2] That is, all paths with positive traffic flow correspond to minimizing latency paths

Optimal flow

Definition 5. *A flow* $\mathbf{f}^*$ *such that*

$$L(\mathbf{f}^*) = \min\{L(\widetilde{\mathbf{f}}) : \widetilde{\mathbf{f}} \in \Delta(\mathscr{P}_{st})\}$$

is referred to as an optimal flow .

Bounding the price of anarchy

Theorem 1. *Let G be a network where one unit of traffic is routed from a source s to a destination t. Let the latency on each edge e be*[3]

[3] Where $a_e, b_e \geq 0$ are constants

$$\ell_e(x) = a_e x + b_e$$

Let $\mathbf{f}$ *be an equilibrium flow and let* $\mathbf{f}^*$ *be an optimal flow. Then*

$$\text{PoA} := \frac{L(\mathbf{f})}{L(\mathbf{f}^*)} \leq \frac{4}{3}$$

Before proving the theorem the following preliminary results must be shown regarding how one transitions between edge and path flows.

Path flows and edge flows

Let $\mathbf{f}_P$ be a path flow. For an edge e the corresponding edge flow is

$$F_e = \sum_{P|e \in P} f_P$$
$$\widetilde{F}_e = \sum_{P|e \in P} \widetilde{f}_P$$

If one considers *all* edges $e \in E$ then it can be seen that a path flow f_P induces an entire *sequence* of *edge flows*

$$\{F_e\}_{e \in E}$$

Proposition 1. *Let* $\mathbf{f}$ *and* $\widetilde{\mathbf{f}}$ *be two path flows with corresponding edge flows*

$$\{F_e\}_{e \in E} \quad , \quad \{\widetilde{F}_e\}_{e \in E}$$

Then[4]

[4] i.e. one can rewrite the total latency in terms of *edge flows* (rather than *path flows*)

$$\sum_P \widetilde{f}_P L_P(\mathbf{f}) = \sum_e \widetilde{F}_e \ell_e(F_e)$$

Proof.

$$\sum_P \widetilde{f}_P L_P(\mathbf{f}) = \sum_P \widetilde{f}_P \left(\sum_{e \in P} \ell_e(F_e) \right)$$

The bracketed sum is a function of the path P, so that for a given path P the coefficient of $\widetilde{f}_P$ is exactly $\sum_{e \in P} \ell_e(F_e)$. However, a given edge e may be shared by *multiple* paths, so that one can first sum over all $\widetilde{f}_P$'s which share the edge e and then multiply by the corresponding factor $\ell_e(F_e)$ – subsequently summing *this* over all edges. By this logic,

$$\sum_P \widetilde{f}_P \left(\sum_{e \in P} \ell_e(F_e) \right) = \sum_e \ell_e(F_e) \left(\sum_{P|e \in P} \widetilde{f}_P \right)$$

But[5]

$$\sum_e \ell_e(F_e)\left(\sum_{P|e\in P} \widetilde{f}_P\right) = \sum_e \widetilde{F}_e \ell_e(F_e)$$

[5] By definition of $\widetilde{F}_e$

□

Corollary 1. *If $\widetilde{\mathbf{f}} = \mathbf{f}$ then by proposition 1*

$$\sum_P f_P L_P(\mathbf{f}) = \sum_e F_e \ell_e(F_e)$$

Lemma 1. *Let $\mathbf{f}$ be a Nash equilibrium flow and let $\widetilde{\mathbf{f}}$ be any other path flow with corresponding edge flows*

$$\{F_e\}_{e\in E} \quad , \quad \{\widetilde{F}_e\}_{e\in E}$$

respectively. Then

$$\sum_e (\widetilde{F}_e - F_e)\ell_e(F_e) \geq 0$$

Proof. By definition of a Nash equilibrium flow

$$f_P > 0 \iff L_P(\mathbf{f}) = \min_{P'\in\mathscr{P}_{st}} L_{P'}(\mathbf{f})$$

Let

$$L := \min_{P'\in\mathscr{P}_{st}} L_{P'}(\mathbf{f})$$

Note that (by definition)

$$\sum_P f_P = \sum_P \widetilde{f}_P = 1$$

Consider the sum

$$\sum_P f_P L_P(\mathbf{f})$$

The terms where $f_P = 0$ can be discarded. If $f_P > 0$ then $L_P(\mathbf{f}) = L$. Therefore, the sum above reduces to

$$\sum_P f_P L_P(\mathbf{f}) = \sum_P f_P \cdot L = L \cdot \sum_P f_P = L \cdot 1 = L$$

Next consider the sum

$$\sum_P \widetilde{f}_P L_P(\mathbf{f})$$

On the other hand, if $\widetilde{f}_P > 0$ one cannot necessarily say[6] that $L_P(\widetilde{f}) = L$. However, by definition of L, it holds that

[6] Because $\widetilde{f}$ is not necessarily a Nash equilibrium flow

$$L_P(\mathbf{f}) \geq L \quad \forall P$$

Then

$$\sum_P \widetilde{f}_P L_P(\mathbf{f}) \geq \sum_P \widetilde{f}_P \cdot L = L \cdot \sum_P \widetilde{f}_P = L \cdot 1 = L$$

Then by proposition 1, corollary 1 and the above two results

$$\begin{aligned}
\sum_e \widetilde{F}_e \ell_e(F_e) &= \sum \widetilde{f}_P L_P(\mathbf{f}) \\
&\geq L \\
&= \sum_P f_P L_P(\mathbf{f}) \\
&= \sum_e F_e \ell_e(F_e) \\
\implies \sum_e (\widetilde{F}_e - F_e)\ell_e(F_e) &\geq 0
\end{aligned}$$

□

Proof of theorem 1

Recall theorem 1

Theorem 1. *Let G be a network where one unit of traffic is routed from a source s to a destination t. Let the latency on each edge e be*[7]

$$\ell_e(x) = a_e x + b_e$$

Let $\mathbf{f}$ *be an equilibrium flow and let* $\mathbf{f}^*$ *be an optimal flow. Then*

$$\text{PoA} := \frac{L(\mathbf{f})}{L(\mathbf{f}^*)} \leq \frac{4}{3}$$

Proof. Let

$$\{F_e^*\}_{e \in E}$$

be the set of edge flows corresponding to the optimal[8] path flow $\mathbf{f}^*$. By lemma 1 [9]

$$L(\mathbf{f}) = \sum_e F_e \ell_e(F_e) = \sum_e F_e(a_e F_e + b_e) \leq \sum_e F_e^*(a_e F_e + b_e)$$

However

$$\begin{aligned}
L(\mathbf{f}) &\leq \sum_e F_e^*(a_e F_e + b_e) \\
\implies L(\mathbf{f}) - L(\mathbf{f}^*) &\leq \sum_e F_e^*(a_e F_e + b_e) - L(\mathbf{f}^*) \\
\implies L(\mathbf{f}) - L(\mathbf{f}^*) &\leq \sum_e F_e^*(a_e F_e + b_e) - \sum_e F_e^* \ell_e(F_e^*) \\
\implies L(\mathbf{f}) - L(\mathbf{f}^*) &\leq \sum_e F_e^*(a_e F_e + b_e) - \sum_e F_e^*(a_e F_e^* + b_e) \\
\implies L(\mathbf{f}) - L(\mathbf{f}^*) &\leq \sum_e F_e^* a_e (F_e - F_e^*)
\end{aligned}$$

Then[10]

$$L(\mathbf{f}) - L(\mathbf{f}^*) \leq \frac{1}{4} \sum_e a_e F_e^2 \tag{1}$$

[7] Where $a_e, b_e \geq 0$ are constants

[8] i.e. *latency minimizing*

[9] Note that in lemma 1 the path flow $\widetilde{\mathbf{f}}$ is *arbitrary*
In this case $\widetilde{\mathbf{f}} = \mathbf{f}^*$ happens to be an optimal flow

[10] Since

$$\left(\frac{x-y}{2}\right)^2 \geq 0 \iff x(y-x) \leq \frac{y^2}{4}$$

where

$$\begin{aligned}
x &= F_e^* \\
y &= F_e
\end{aligned}$$

But[11]

$$L(\mathbf{f}) = \sum_e F_e(a_e F_e + b_e) \geq \sum_e F_e(a_e F_e) = \sum_e a_e F_e^2$$

Combining this with (1) results in

$$L(\mathbf{f}) - L(\mathbf{f}^*) \leq \frac{1}{4} L(\mathbf{f})$$
$$\implies \frac{3}{4} L(\mathbf{f}) \leq L(\mathbf{f}^*)$$
$$\implies \text{PoA} := \frac{L(\mathbf{f})}{L(\mathbf{f}^*)} \leq \frac{4}{3}$$

□

[11] Since $b_e \geq 0$

Corollary 2 (Adding roads)**.** *If additional roads are added to a road network with affine latency functions, the latency at Nash equilibrium can increase by at most a factor of* $4/3$*.*

Proof. Let G be a road network and let H be an augmented version[12] of G. Let $\mathbf{f}_G$ and $\mathbf{f}_H$ denote equilibrium flows in networks G and H, respectively. Similarly, let $\mathbf{f}_G^*$ and $\mathbf{f}_H^*$ denote optimal flows in networks G and H, respectively. Note[13]

$$L(\mathbf{f}_H^*) \leq L(\mathbf{f}_G^*)$$

Then[14] by theorem 1

$$L(\mathbf{f}_H) \leq \frac{4}{3} L(\mathbf{f}_H^*) \leq \frac{4}{3} L(\mathbf{f}_G^*) \leq \frac{4}{3} L(\mathbf{f}_G)$$

□

[12] Added roads

[13] Since in general

$$S \subset T \implies \min T \leq \min S$$

[14] i.e. when the latency function $\ell_e(x)$ is affine, adding roads can worsen total latency by at most a factor of $4/3$ in the Nash equilibrium-case

3.5 *Pigou networks and general latency bounds*

Pigou networks

Let s and t be nodes in a network G. Suppose that there is a fixed number of drivers r traveling from s to t. A *Pigou network* between nodes s and t is a set of two paths from s to t with corresponding latency functions $\ell(x)$ and $\ell(r)$. In a Pigou network x drivers travel along the route with corresponding latency function $\ell(x)$ and $r - x$ drivers travel along the route with corresponding latency function $\ell(r)$ (where $\ell(r)$ is a latency function that is *independent* of traffic). The total latency between nodes s and t under this regime is then

$$x \cdot \ell(x) + (r - x) \cdot \ell(r)$$

The remarkable fact is that the price of anarchy of *any* class of latency functions $\mathscr{L}$ on a given network is maximized in a Pigou network.

3.6 General latency bound

Theorem 2. *Let $\mathscr{L}$ be any class of latency functions. Let G be a network with latency functions in $\mathscr{L}$ and total flow r from node s to node t. Then*

$$\text{PoA} \leq A_r(\mathscr{L})$$

where[15]

$$A_r(\mathscr{L}) := \max_{0 \leq \tilde{r} \leq r} \sup_{\ell \in \mathscr{L}} \alpha_{\tilde{r}}(\ell)$$

$$\alpha_r(\ell) := \frac{r \cdot \ell(r)}{\min_{0 \leq x \leq r}[x \cdot \ell(x) + (r-x) \cdot \ell(r)]}$$

[15] Note that the numerator is the event in which *all* drivers (r units) take the road on which the latency is independent of the traffic — this corresponds to a worst-case Nash equilibrium flow

For this reason, $\alpha_r(\ell)$ is referred to as the Pigou price of anarchy — whereby the denominator is the optimal latency over all traffic distributions in a Pigou network

Remark 1. *By definition,*

$$\min_{0 \leq x \leq r}[x \cdot \ell(x) + (r-x) \cdot \ell(r)] \leq x \cdot \ell(x) + (r-x) \cdot \ell(r) \quad \forall x \in [0, r]$$

In fact, the minimum can be taken over all $x \geq 0$. To show this first note that latency functions are weakly increasing by assumption. Therefore[16]

[16] Note that $x \geq 0$ *integer-valued*

$$\begin{aligned} x \geq r &\implies \ell(x) \geq \ell(r) \\ &\implies x \cdot \ell(x) \geq x \cdot \ell(r) \\ &\implies x \cdot \Big(\ell(x) - \ell(r)\Big) \geq 0 \\ &\implies x \cdot \ell(x) + (r-x) \cdot \ell(r) = x \cdot \Big(\ell(x) - \ell(r)\Big) + r \cdot \ell(r) \geq r \cdot \ell(r) \end{aligned}$$

and

$$x \cdot \ell(x) + (r-x) \cdot \ell(r) = x \cdot \Big(\ell(x) - \ell(r)\Big) + r \cdot \ell(r) \geq r \cdot \ell(r)$$

On the other hand

$$\begin{aligned} 0 \leq y \leq r &\implies \ell(y) \leq \ell(r) \\ &\implies \ell(y) - \ell(r) \leq 0 \end{aligned}$$

Therefore

$$0 \leq y \leq r \implies y \cdot \ell(x) + (r-y) \cdot \ell(r) = y \cdot \Big(\ell(y) - \ell(r)\Big) + r \cdot \ell(r) \leq r \cdot \ell(r)$$

If one defines

$$f(x) := x \cdot \ell(x) + (r-x) \cdot \ell(r)$$

Then what has been shown is that

$$x \geq r \implies f(x) \geq f(y) \quad \forall y \in [0, r]$$

Therefore

$$\min_{0 \leq x \leq r} f(x) = \min_{x \geq 0} f(x)$$

As a corollary

$$\min_{0\le x\le r}[x\cdot \ell(x)+(r-x)\cdot \ell(r)] \le x\cdot \ell(x)+(r-x)\cdot \ell(r)$$

for any $x \ge 0$. *Or*

$$\frac{1}{\min_{0\le x\le r}[x\cdot \ell(x)+(r-x)\cdot \ell(r)]} \ge \frac{1}{x\cdot \ell(x)+(r-x)\cdot \ell(r)}$$

for any $x \ge 0$.

Proof. Let $\mathbf{f}$ and $\mathbf{f}^*$ be an equilibrium and optimal flow in G, respectively, with corresponding edge flows $\mathbf{F}$ and $\mathbf{F}^*$. Fix an edge $e \in G$ and consider a Pigou network with latency $\ell(x) := \ell_e(x)$ and total flow $r := F_e$. Then[17]

$$\begin{aligned}
\alpha_{F_e(\ell_e)} &= \frac{F_e\cdot \ell_e(F_e)}{\min_{0\le x\le F_e}[x\cdot \ell_e(x)+(F_e-x)\cdot \ell_e(F_e)]}\\
&\ge \frac{F_e\cdot \ell_e(F_e)}{F_e^*\cdot \ell_e(F_e^*)+(F_e-F_e^*)\cdot \ell_e(F_e)}\\
&\implies F_e^*\ell_e(F_e^*) \ge \frac{1}{\alpha_{F_e}(\ell_e)}F_e\cdot \ell_e(F_e)\cdot \ell_e(F_e)+(F_e^*-F_e)\cdot \ell_e(F_e)\\
&\implies L(\mathbf{f}^*) \ge \frac{1}{A_r(\mathscr{L})}\cdot L(\mathbf{f})+\sum_e (F_e^*-F_e)\cdot \ell_e(F_e) \ge \frac{1}{A_r(\mathscr{L})}L(\mathbf{f})
\end{aligned}$$

By *lemma 1*

□

[17] By *remark 1*
Note that the penultimate step holds by taking the sum over all edges e

3.7 *Doubling traffic*

IT IS A REMARKABLE FACT that the total latency of a Nash-equilibrium flow can be *no worse* than the latency of an optimal flow with *double* the number of drivers.

Theorem 3. *Let G be a road network with a specified source s and sink t where r units of traffic are routed from s to t. Let $\mathbf{f}$ be a Nash equilibrium flow with total latency $L(\mathbf{f})$. Let $\mathbf{f}^*$ be an optimal flow when $2r$ units of traffic are routed in the same network, with corresponding total latency $L(\mathbf{f}^*)$. Then*

$$L(\mathbf{f}) \le L(\mathbf{f}^*)$$

Remark 2. *The following inequality is required for the proof below.*

$$\sum_P f_P^* L_P(\mathbf{f}) \le L(\mathbf{f}) + L(\mathbf{f}^*)$$

which, when rewritten in terms of edges, is equivalent to

$$\sum_e F_e^*\ell_e(F_e) \le \sum_e F_e^*\ell_e(F_e^*) + \sum_e F_e\ell_e(F_e)$$

To show this it is enough to show that

$$F_e^* \Big(\ell_e(F_e) - \ell_e(F_e^*) \Big) \leq F_e \ell_e(F_e)$$

for all edges $e \in E$. But this just follows from the fact that latency functions are by assumption weakly increasing .

$$\begin{aligned} F_e < F_e^* &\implies \ell_e(F_e) - \ell_e(F_e^*) \leq 0 \\ &\implies F_e^* \cdot \Big(\ell_e(F_e) - \ell_e(F_e^*) \Big) \leq 0 \leq F_e \ell_e(F_e) \end{aligned}$$

$$\begin{aligned} F_e^* \leq F_e &\implies \ell_e(F_e) - \ell_e(F_e^*) \\ &\implies F_e^* \cdot \Big(\ell_e(F_e) - \ell_e(F_e^*) \Big) \leq F_e \cdot \Big(\ell_e(F_e) - \ell_e(F_e^*) \Big) \leq F_e \cdot \ell_e(F_e) \end{aligned}$$

where the last inequality holds by the non-negativity of the difference.

Proof. Let $L := \min_{P \in \mathscr{P}_{st}} L_P(\mathbf{f})$. Then[18] $f_P > 0 \iff L_P(\mathbf{f}) = L$. As in *lemma 1*

[18] By definition of a Nash equilibrium flow

$$L(\mathbf{f}) = \sum_P f_P L_P(\mathbf{f}) = \sum_P f_P \cdot L = L \cdot \sum_P f_P = L \cdot r$$

and (by the same reasoning as in lemma 1)

$$\sum_P f_P^* L_P(\mathbf{f}) \geq 2r \cdot L$$

Then

$$\begin{aligned} L(\mathbf{f}^*) = \sum_P f_P^* L_P(\mathbf{f}) &\geq 2r \cdot L \\ &= 2 \cdot L(\mathbf{f}) \end{aligned}$$

Combining this with *remark 1* results in

$$\begin{aligned} 2 \cdot L(\mathbf{f}) &\leq L(\mathbf{f}) + L(\mathbf{f}^*) \\ \implies L(\mathbf{f}) &\leq L(\mathbf{f}^*) \end{aligned}$$

□

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