The weak law of large numbers

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Below is a brief discussion on the weak law of large numbers, a very standard result in probability. I like the proof because of its brevity. The statement of the theorem is as follows.

Convergence in probability

Let $X_1, X_2, \ldots$ be a sequence of *iid* random variables. Define

$$p_n(\epsilon) := \mathrm{P}\big(|X_n - X| < \epsilon\big)$$

$|X_n - X| < \epsilon$ is the event that X_n deviates from the random variable X in magnitude by not more than ϵ. $p_n(\epsilon)$ is the probability of such an event.

Let $\delta \in (0, 1)$ and let $\epsilon > 0$ be given.

Convergence in probability means that there exists an N such that

$$n > N \implies p_n(\epsilon) \geq 1 - \delta.$$

X_n is then said to converge to X _in probability*. This is denoted as

$$X_n \xrightarrow{p} X.$$

In words, if one wants to permit X_n to deviate from X by less than an $\epsilon-$margin with at least $[(1-\delta)\cdot 100]\%$ certainty, there will always exist an N which achieves this *for all* $n > N$ (assuming the X_n's are *iid*).

Let X be a random variable and let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. copies of X. Define

$$\overline{X_n} := \frac{\sum_{i=1}^{n} X_i}{n}.$$

Then,

$$\overline{X_n} \xrightarrow{p} \mu := \mathrm{E}[X].$$

The proof

The proof hinges on the well-known tail-bound,

$$\mathrm{P}(h(X) \geq a) \leq \frac{\mathrm{E}h(X)}{a}.$$

Where $h \geq 0$.

Let $X = \overline{X_n}$ and $h(\overline{X_n}) = (\overline{X_n} - \mu)^2$.

Then,

$$\begin{aligned}
\mathrm{P}\big(|\overline{X_n} - \mu| < \epsilon\big) &= \mathrm{P}\big((\overline{X_n} - \mu)^2 < \epsilon^2\big) \\
&= 1 - \mathrm{P}\big((\overline{X_n} - \mu)^2 \geq \epsilon^2\big) \\
&\geq 1 - \frac{\mathrm{E}(\overline{X_n} - \mu)^2}{\epsilon^2} \\
&= 1 - \frac{1}{n} \cdot \frac{\sigma^2}{\epsilon^2}
\end{aligned}$$

Note the use of the i.i.d. assumption in the penultimate step where $\mathrm{Var}\overline{X_n} = \frac{\sigma^2}{n}$.

The term

$$-\frac{1}{n} \cdot \frac{\sigma^2}{\epsilon^2}$$

must be bounded below by $-\delta$ in order to obtain the desired inequality

$$1 - \frac{1}{n} \cdot \frac{\sigma^2}{\epsilon^2} \geq 1 - \delta.$$

The only factor free to be altered is n and

$$-\frac{1}{n} \cdot \frac{\sigma^2}{\epsilon^2} \geq -\delta \iff n > \frac{\sigma^2}{\epsilon^2 \delta}.$$

Application

Operationally then, given ϵ, δ, and σ^2, the weak law of large numbers tells you how large n needs to be in order to fall within ϵ of X with probability $1 - \delta$.