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CALDERÒN REVISITED — A SYNOPSIS

A PAPER SUBMITTED FOR *TOPICS IN GEOMETRIC ANALYSIS: GEOMETRIC INVERSE PROBLEMS* UNDER THE GUIDANCE OF DR ALEXAKIS

Contents

Abstract

A synopsis of Calderón's paper *On an Inverse Boundary Problem* is provided.

A Statement of the Problem

Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary ∂D
The electrical conductivity of D is represented by a function $\gamma(x)$.
Let w represent the induced potential.
By Ohm's law the current through a conductor between two points is proportional to the potential–with the constant of proportionality equal to the inverse of the resistance.
Therefore, $\gamma \nabla w$ represents the *current flux*.
If no current is created or destroyed, and ϕ represents the potential on the boundary of D, then:

$\gamma(x)$ is assumed to have a positive lower bound.
In practice γ is a matrix—and under that condition this is still an open problem.

Note that the potential γ varies inversely as the resistance. Ohm's law can then be rephrased as stating that the current is proportional to the product of the *conductivity* and the potential.

$$\begin{aligned} \nabla \cdot (\gamma \nabla w) &= 0 \\ w|_{\partial D} &= \phi \end{aligned} \tag{1}$$

Sobolev spaces

The following conditions are normally imposed:

$$\begin{aligned} \phi &\in H^{\frac{1}{2}}(\partial D) \\ w &\in H^1(D) \end{aligned}$$

The intuition behind these conditions is oftentimes one is interested in conductivities that are bounded but not necessarily continuous.
The first condition is related to the *change-of-variables strategy*.
Suppose that (1) holds.
Let v be a function such that $v|_{\partial D} = \phi$.
Let $\widetilde{u} = w - v$.
Then:

Some of the most interesting applications of this inverse problem are in *electrical impedance tomography* in which case the conductivities present in muscle activation may be *discontinuous*

$$\begin{aligned} \nabla \cdot (\gamma \nabla \widetilde{u}) &= -\nabla \cdot (\gamma \nabla v) \\ \widetilde{u}|_{\partial D} &= 0 \end{aligned} \tag{2}$$

By standard functional analysis, if $\phi \in H^{\frac{1}{2}}(\partial D)$ then there exists a $v \in H^1(D)$ such that $v|_{\partial D} = \phi$.
Then (2) will always have a solution in H^1 .

Note that this allows us to minimize the total number of assumptions made on the functions of interest—it also allows a wider class of conductivities to be considered.

The second condition [1] is related to the notion of a *weak solution* . Suppose that (1) holds and let $\varphi \in C_c^\infty$ such that $\varphi, \varphi' = 0$ on ∂D. Then:

$$\int_D \nabla \cdot (\gamma \nabla w)\varphi = 0 - \int \gamma \nabla w \cdot \nabla \varphi = 0$$
$$\iff -\int \gamma \nabla w \cdot \nabla \varphi = 0 \tag{3}$$

Then we say that w is a *weak* solution if it satisfies (3) and $w = \phi$ on ∂D

[1] $w \in H^1(D)$

One can think of φ as a "test" function.

Note that conditions are put on φ only up to φ' despite the fact that *two* derivatives are taken in $\nabla \cdot (\gamma \nabla w)$.

Dirichlet-Neumann map

The Dirichlet to Neumann map (or voltage to current map) is given by:

$$\Lambda_\gamma(\phi) = \left(\gamma \frac{\partial w}{\partial \nu}\right)\Bigg|_{\partial D}$$

ν here represents the outer unit normal to ∂D

Note that this is *non-linear* in γ

THE QUESTION IS: does $\Lambda_\gamma(\phi)$ reconstruct γ? It is generally difficult to prescribe a voltage measurement ϕ at the boundary in a way that allows the conductivity to be determined easily. Calderón instead made the following observation:

$$\begin{aligned}\int_{\partial D} \phi \Lambda_\gamma(\phi) &= \int_{\partial D} w\gamma \nabla w \cdot \nu \\ &= \int_D \nabla \cdot \Big((\gamma \nabla w) w\Big) \\ &= \int_D \underbrace{\nabla \cdot (\gamma \nabla w)}_{=0} w + \underbrace{\int_D \gamma \nabla u \cdot \nabla u}_{\text{energy/power}} \\ &= \int_D \gamma (\nabla w)^2\end{aligned} \tag{4}$$

Note the use of both the divergence theorem and the product rule with respect to the divergence operator. dS here denotes the surface measure and w represents the solution of (1).

According to the relationship established in (4) it can then be seen that to know $\Lambda_\gamma(\phi)$ and to know $Q_\gamma(\phi)$ for all $\phi \in H^{\frac{1}{2}}(\partial D)$ is equivalent.

One therefore *defines*:

$$Q_\gamma(\phi) := \int_D \gamma (\nabla w)^2 \, dx$$

$Q_\gamma(\phi)$ measures the energy needed to maintain the potential ϕ at the boundary.

Note that Q_γ is a function of ϕ because of the dependence of w on ϕ.

From the point of view of $Q_\gamma(\phi)$ the problem can then be interpreted as needing to find enough solutions $w \in H^1(D)$ of (1) in order to find γ in the interior.[2]

[2] This problem was solved in the general case almost 30 years later by Uhlmann for the case $n \geq 3$
Ironically, the case $n = 2$ was more difficult to solve and required high-frequency analysis
The kernel of Λ is a function of $2(n-1)$ variables which, in general, is $\geq n$ (i.e. the problem is *over-determined*)
In the case that $n = 2$ it happens to be that $2(n-1) = n$ which means that one has *just enough* data to solve the problem

Therefore consider the map:

$$\Phi : \gamma \longrightarrow Q_\gamma$$

Where $Q_\gamma = \{Q_\gamma(\phi) : \phi \text{ Dirichlet data } \}$. Note that Φ is a *functional.*

The *function* γ is mapped to the *number* $Q_\gamma(\phi)$ for some Dirichlet $\phi \in H^{\frac{1}{2}}(\partial D)$

The Setup

Strategy

It has been established that to know $Q_\gamma(\phi)$ and to know $\Lambda_\gamma(\phi)$ is equivalent. The strategy is to:

- Impose a norm on the space Q_γ to construct a well-defined notion of Φ' (in the linearized case)
- Show that Φ' is injective (in the linearized case) thereby proving (via the *inverse function theorem*) the local invertibility of the linearized version of Φ

Recall the inverse function theorem which states that if the derivative of a function at one point is injective, then the map is at least locally invertible.

Functional Derivatives

Variation in the function $\gamma : D \to \mathbb{R}$ is represented as:

$$\Delta\gamma = \epsilon \cdot \eta(x)$$

for some η.
Fix γ_0 .
Variation in Φ (with respect to γ_0 is then represented as:

$$\Delta\Phi := \Phi(\gamma_0 + \Delta\gamma) - \Phi(\gamma_0)$$

Note that the functional $\Phi(\gamma_0 + \epsilon \cdot \eta)$ is an ordinary function of ϵ. It therefore has a standard Taylor expansion in terms of powers of ϵ as follows:

$$\begin{aligned}\Phi(\gamma_0 + \epsilon \cdot \eta) &= \Phi(\gamma_0) + \left.\frac{d\Phi(\gamma_0 + \epsilon \cdot \eta)}{d\epsilon}\right|_{\epsilon=0} \cdot \epsilon + \frac{1}{2}\left.\frac{d^2\Phi(\gamma_0 + \epsilon \cdot \eta)}{d\epsilon^2}\right|_{\epsilon=0} \cdot \epsilon^2 + \cdots \\ &= \sum_{n=0}^{N} \frac{1}{n!}\left.\frac{d^n\Phi(\gamma_0 + \epsilon \cdot \eta}{d\epsilon^n}\right|_{\epsilon=0} \cdot \epsilon^n + O\left(\epsilon^{N+1}\right)\end{aligned}$$

Calderòn considers the case:

$$\gamma_0 \equiv 1$$

and absorbs ϵ by using the notation
$\delta(x) := \epsilon \cdot \eta(x)$
This is a *linearized* version of the problem (Linearized inverse conductivity problem) in which case we deal with the linear approximation:

$$\Phi(1+\delta) \approx \Phi(\gamma_0) + \left.\frac{d\Phi(1+\delta)}{d\epsilon}\right|_{\epsilon=0} \cdot \epsilon$$

By definition, for given Dirichlet data ϕ,

$$\Phi(1) = Q_1(\phi) = \int_D 1 \cdot (\nabla w)^2 = \int_D (\nabla w)^2$$

Which is completely known. What is *unknown* is $d\Phi/d\epsilon$.

Making sense of $d\Phi/d\epsilon$

By definition:

$$\frac{d\Phi}{d\epsilon} = \lim_{\epsilon\to 0} \frac{\Phi(1+\delta) - \Phi(1)}{\epsilon}$$

To take limits we need an appropriate *norm*.
However, note that, given Dirichlet data ϕ:

Recall that Φ is an ordinary function in ϵ.
Recall that $\delta := \epsilon \cdot \eta$ is an implicit function of ϵ

$$\Phi(1+\delta) - \Phi(1) = Q_{1+\delta}(\phi) - Q_1(\phi)$$

Therefore it is enough to consider a norm on the space Q_γ with respect to Dirichlet data ϕ.
Such a norm would be:

$$||\phi||^2 = \int_D |\nabla u|^2 \, dx$$
$$||Q_\gamma|| = \sup_{||\phi||\leq 1} |Q_\gamma(\phi)|$$

Where $u\Big|_{\partial D} = \phi$ and $\nabla u = 0$ in D.
This is a very natural "Sobolev" norm to use

With this norm Φ is bounded and analytic in the set:

$$S = \{ \text{ real-valued } f \text{ with positive lower bound } \} \subset L^\infty$$

Let w be a solution to (1) when $\gamma = 1+\epsilon$ and let u be a solution to (1) in the particular case where $\gamma = 1$
Letting $v = w - u$ the expression $w = u + v$ gives an expression for a solution to (1) for $\gamma = 1+\delta$. This expression can be thought of as the solution u to (1) when $\gamma = 1$ plus an "error" solution.
The idea is that since u is known only v needs to be known to find the solution w .
Let L_γ be the differential operator:

i.e.
$\nabla \cdot (1 \cdot \nabla u) = 0$ and $u\Big|_{\partial D} = \phi$
$\nabla \cdot \Big((1+\delta)\cdot \nabla w\Big) = 0$ and $w\Big|_{\partial D} = \phi$

$$L_\gamma(f) = \nabla \cdot (\gamma \nabla f)$$

i.e. L_γ is just the conductivity equation

Applying this differential operator to w for $\gamma = 1 + \epsilon$ results in

$$\begin{aligned} L_\gamma(w) &= L_\gamma u + L_\gamma v \\ &= L_{1+\delta} u + L_{1+\delta} v \\ &= L_1 u + L_\delta u + L_1 v + L_\delta v \\ &= 0 + L_\delta u + L_1 v + L_\delta v \\ &= L_1 v + L_\delta v + L_\delta u \qquad (5) \\ &= 0 \end{aligned}$$

If L_1 is interpreted as an operator $L_1 : H_0^1(D) \to H^{-1}(D)$ then it has a bounded inverse.
The key assumption is that u and w have equal Dirichlet data ϕ on ∂D . That way $v \in H_0^1(D)$.
Then L_1 has a bounded inverse G.
Applying G to (5) results in

$H_0^1(D)$ is the closure in $H^1(\mathbb{R}^n)$ of functions of C^∞ with support in D

$$v + GL_\delta v = v(1 + GL_\delta) = -GL_\delta u$$

Then

$$v = -\left[\sum_{j=1}^{\infty} (-1)^j (GL_\delta)^j\right] (GL_\delta u) \qquad (6)$$

Using the geometric series formula for matrices
G is linear and has a corresponding matrix

Known

$$w \in H_0^1(D) \implies ||L_\delta w||_{H^{-1}(D)} \leq ||\epsilon||_{L^\infty} ||w||_{H_0^1(D)}$$

Let $A = ||G||$.
the above series converges for $||\delta||_{L^\infty} A < 1$ and

$$||v||_{H^1(D)} \leq \frac{||\delta||_{L^\infty} ||\phi||}{1 - A||\delta||_{L^\infty}} \qquad (7)$$

Sketch of derivation of convergence condition

It is known that $||GL_\delta|| \leq ||\delta||_{L^\infty} \cdot A$.
By the triangle inequality it follows from (6) and the above bound on $||GL_\delta||$ that

$$\begin{aligned} ||v||_{H^1(D)} &\leq \sum_{j=1}^{\infty} ||GL_\delta||^j ||GL_\delta u|| \\ &\leq \sum_{j=1}^{\infty} (A \cdot ||\delta||_{L^\infty})^j ||GL_\delta u|| \\ &\leq \frac{1}{1 - A \cdot ||\delta||_{L^\infty}} \cdot ||GL_\delta u|| \\ &\leq \frac{1}{1 - A \cdot ||\delta||_{L^\infty}} \cdot ||\delta||_{L^\infty} ||\phi|| \end{aligned}$$

This is why we need $A \cdot ||\delta||_{L^\infty} < 1$ everywhere.

In general $||Gx|| \leq M||x||$
The *least* such number is $||G||$.

Calderòn's Seminal Idea

Computing $dQ_\gamma(\phi)\Big|_{\gamma=1} = Q_{1+\delta}(\phi) - Q_1(\phi)$

$$
\begin{aligned}
&Q_{1+\delta}(\phi) - Q_1(\phi)\\
&= \int_D (1+\delta)|\nabla w|^2\, dx\\
&= \int_D \Big[(1+\delta)|\nabla u|^2 + 2(\nabla u \cdot \nabla u) + 2\delta(\nabla u \cdot \nabla v) + (1+\delta)|\nabla v|^2 - |\nabla u|^2\, dx\\
&= \int_D \delta|\nabla u|^2\, dx
\end{aligned}
$$

Want to show that

$$\int_D \delta|\nabla u|^2\, dx = 0 \ \ \forall u \implies \delta = 0$$

The injectivity of $dQ_\gamma(\phi)\Big|_{\gamma=1}$ then follows. [3]
It suffices to show that

$$\int_D \delta(\nabla u_1 \cdot \nabla u_2)\, dx = 0$$

for any harmonic functions u_1, u_2 .

[3] A is invertible $\iff Ax = 0 \implies x = 0$
Note that this can be interpreted as the Fréchet derivative of Q at $\gamma = \gamma_0$ in the direction $\delta(x)$

Lemma.

$$\int_D \delta|\nabla u|^2\, dx = 0 \implies \int_D \delta(\nabla u_1 \cdot \nabla u_2)\, dx = 0 \ \ \forall u_1, u_2 \text{ harmonic}$$

Proof. This follows from the following interpolation argument.
Let u_1, u_2 be harmonic and suppose the above hypothesis holds.
Note that u_1, u_2 harmonic $\implies u_1 + u_2$ harmonic

$$
\begin{aligned}
&\int \delta(x)|\nabla(u_1+u_2)|^2\\
&= \int \delta|\nabla u_1|^2 + \int \delta|\nabla u_2|^2 + 2\int \delta\nabla u_1 \cdot \nabla u_2 = 0\\
&\implies \int \delta(x)\nabla u_1 \cdot \nabla u_2 = 0
\end{aligned}
$$

□

It seems like a trivial consequence but the above result reduces the problem to whether the product of gradients of harmonic functions is dense.
This is easily the most important part of the paper.
The question that remains from the above definition is what one should choose for u_1 and u_2?
These were Calderòn's choices:

$$\begin{aligned} u_1 &= e^{x\cdot\rho} \\ u_2 &= e^{-x\cdot\overline{\rho}} \end{aligned} \tag{8}$$

with the condition that $\rho \cdot \rho = 0$ ($\rho \in \mathbb{C}^n$) .
The condition that $\rho \cdot \rho = 0$ ensures that u_1, u_2 are harmonic . [4]
Also, $\rho \in \mathbb{C}$ ensures that we obtain non-trivial such ρ's .
Note that the above condition is equivalent to

$$\begin{aligned} \rho &= \frac{x+iy}{2} \qquad x, y \in \mathbb{R}^n \\ |x| &= |y|, \quad x \cdot y = 0 \end{aligned}$$

[4] e.g. for u_1 if $\rho \cdot \rho = 0$ then $\nabla \cdot (\nabla \cdot u_1) = \rho \cdot \rho e^{x\cdot\rho} = 0 \cdot e^{x\rho} = 0$
For $x, y \in \mathbb{R}$ $\quad x \cdot y = 0 \iff x$ or $y = 0$

In general for $z = x + iy \in \mathbb{C}$

$$\begin{aligned} z \cdot z = 0 &\iff (x+iy)\cdot(x+iy) \\ &\iff \begin{cases} |x|^2 = |y|^2 \\ x \cdot y = 0 \end{cases} \end{aligned}$$

The upshot of this choice for u_1, u_2 is what happens when one computes $\int_D \delta(\nabla u_1 \cdot \nabla u_2)$.

$$\begin{aligned} \int_D \delta(\nabla u_1 \cdot \nabla u_2)\, dx &= \int_D \delta(\nabla e^{x\cdot\rho} \nabla e^{-x\cdot\rho})\, dx \\ &= -|\rho|^2 \int_D \delta \cdot e^{x\cdot(\rho-\overline{\rho})}\, dx \\ &= -|\rho|^2 \cdot \widehat{\delta}(\rho - \overline{\rho}) = 0 \end{aligned}$$

Note that $\rho - \overline{\rho}$ is purely imaginary and $\widehat{\delta}$ denotes the Fourier transform of δ

By the *Fourier inversion theorem*

$$\widehat{\delta} = 0 \implies \delta \equiv 0$$

Estimating γ

Let $\widehat{\gamma}$ denote the Fourier transform of γ .[5]

The goal in this section is to estimate $\widehat{\gamma}$ as a means of estimating γ .

To do that one should write $\widehat{\gamma}$ as the sum of something known and an error term.

The following bilinear form is *completely known* for dirichlet data ϕ_1, ϕ_2 and solutions $w_1 = u_1 + v_1, w_2 = u_2 + v_2$.

$$B(\phi_1, \phi_2) = \frac{1}{2}\Big[Q_\gamma(w_1 + w_2) - Q_\gamma(w_1) - Q_\gamma(w_2)\Big]$$

Let ρ be as in (8).

Let

$$\widehat{F}(\rho) = \frac{1}{|\rho|^2} B(e^{x\cdot\rho}, e^{-x\cdot\bar{\rho}})$$

Then

$$\widehat{\gamma}(\rho) = \widehat{F}(\rho) + R(\rho)$$

It follow from (7) that

$$|R(\rho) \leq C \cdot ||\delta||^2_{L^\infty} e^{2\pi r\cdot|\rho|}$$

provided that $A \cdot ||\delta||_{L^\infty} \leq 1 - \epsilon$.[6]

If one chooses $\alpha \in (1, 2)$ such that

$$|\rho| \leq \frac{2-\alpha}{2\pi r} \log \frac{1}{||\delta||_{L^\infty}} = \sigma$$

then

$$|R(\rho) \leq C \cdot ||\delta||^\alpha_{L^\infty} \tag{9}$$

Let η be a function such that

1. $\widehat{\eta} \in C^\infty$

2. $\text{supp}\widehat{\eta} \subset \{|\rho| \leq 1\}$

3. $\widehat{\eta}(0) = 1$

[5] Extended to be 0 on ∂D .

[6] Where C depends only on D and ϵ and r is the radius of the smallest sphere containing D

Let $\eta_\sigma = \sigma^n \eta(\sigma\rho)$.
Then

$$\widehat{\gamma}\widehat{\eta}\left(\frac{\rho}{\sigma}\right) = \widehat{F}(\rho)\widehat{\eta}\left(\frac{\rho}{\sigma}\right) + R(\rho)\widehat{\eta}\left(\frac{\rho}{\sigma}\right) \tag{10}$$

7

The idea is that $\widehat{\gamma}\widehat{\eta}$ is a good approximation to $\widehat{\gamma}$.
The Fourier transforms in (9) can be obtained by taking the Fourier transform of the convolutions in the following expression.

[7] $\widehat{\eta}$ is just the Fourier inverse of the bump function
The idea is to kill all frequencies greater than σ
η can be thought of as similar to the δ function

$$(\gamma * \eta_\sigma)(x) = (F * \eta_\sigma)(x) + \tau(x) \tag{11}$$

$\gamma * \eta_\sigma$ can be obtained rom the DM map

The size of $\tau(x)$ is just the $L^\infty-$norm which is bounded by the L^1-norm of its Fourier transform .
Then

$$\begin{aligned}
|\tau| &= ||\tau||_{L^\infty} \\
&\leq ||\widehat{\tau}||_1 \\
&= ||R(\rho)\eta||_1 \\
&\leq C \cdot ||\delta||^{\alpha}_{L^\infty} \int \left|\widehat{\eta}\left(\frac{\rho}{\sigma}\right)\right| \\
&= C_1 ||\delta||^{\alpha}_{L^\infty} \left[\log \frac{1}{||\delta||_{L^\infty}}\right]^n
\end{aligned}$$

Where C_1 depends only on $D, \alpha,$ and ϵ

Therefore for $||\delta||_{L^\infty}$ sufficiently small (11) gives an approximation of $\gamma * \eta_\sigma$ with an error much smaller than $||\delta||_{L^\infty}$
If $||\delta||_{L^\infty}$ is small then σ is large and $\gamma * \eta_\sigma$ is a good approximation of γ .