Topics in Ergodic Theory: Introduction to Random Walks on Groups

Giulio Tiozzo

Contents

0 Acknowledgements

These notes have been written for a graduate course in Toronto in Fall 2016. We thank Robbie Nohra for typing the notes. We also thank Malors Espinosa Lara and Yvon Verberne for their help.

1 Introduction and General Setup

Definition 1.1. Let G be a group and (X, d) a metric space. The *isometry group of* X is the group of elements which preserve distance:

$$\text{Isom}(X) = \{f : X \to X : d(x, y) = d(f(x), f(y)) \text{ for all } x, y \in X\}$$

Definition 1.2. A *group action* of G on X is a homomorphism

$$\rho : G \to \text{Isom}(X).$$

Example 1.1. $\mathbb{R}$ acting on itself by translations

$$\rho : \mathbb{R} \to \text{Isom}\mathbb{R}$$
$$\rho(t) : x \mapsto x + t$$

Let now consider μ a probability measure on G. Then we can define a sequence (x_n) of independent random elements of G with distribution μ . The sequence (x_n) is called the sequence of *increments*, and we are interested in the products

$$g_n := x_1 \dots x_n$$

The sequence (g_n) is called a *sample path* for the random walk. If you fix some a basepoint $x \in X$ (where X is the metric space) you can look at the sequence $(g_n \cdot x) \subseteq X$.

Example 1.2. The group $G = \mathbb{Z}$ acts by translations on $X = \mathbb{R}$. Let us take $\mu = \frac{\delta_{+1}+\delta_{-1}}{2}$, i.e. one moves forward by 1 with probability $\frac{1}{2}$ and moves backward by 1 with probability $\frac{1}{2}$.

Example 1.3. $X = O, \quad G = \mathbb{R}^d$

Example 1.4. X = 4-valent tree, $\quad G = \mathbb{F}_2 =$ reduced word in alphabet a, b, a^{-1}, b^{-1} . *Reduced means no redundant pairs. i.e.*

- no a after a^{-1}
- no a^{-1} after a
- no b after b^{-1}
- no b^{-1} after b

$$\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$$

Example 1.5. $S = S^{-1}$ a finitely generating set. $\Gamma = \text{Cay}(G, S)$ is a graph (Cayley Graph) whose vertices are the elements of G and there is an edge from $g \to h$. $(g, h \in G)$ if $h = gs$ where $s \in S$. Let $G = \mathbb{F}_2$, $S = \{a, b\}$, $\text{Cay}(\mathbb{F}_2, S) =$ 4-valent tree . Now let $G = \mathbb{Z}^2$, $S = \{(1,0),(0,1)\}$ then $\text{Cay}(\mathbb{Z}^2, S) =$ grid . $\text{Cay}(\mathbb{Z}^2, S)$ has loops .

Example 1.6.

$$(0,1) + (-1,0) + (0,-1) + (1,0) = (0,0)$$

We wouldn't have a corresponding loop in $\mathbb{F}_2$ as the element in $\mathbb{F}_2$ that corresponds to the loop is $ab^{-1}a^{-1}b$.

Example 1.7. Consider $G = SL_2(\mathbb{R}) = \{A \in M_2 : \det A = 1\}$ which acts on the hyperbolic plane $X = \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

The group G acts by isometries for the *hyperbolic metric* $ds = \frac{dx}{y}$. Let $A, B \in SL_2(\mathbb{R}), \mu = \frac{1}{4}(\delta_A + \delta_{A^{-1}} + \delta_B + \delta_{B^{-1}})$ The boundary of $\mathbb{H}$ is $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. As we will see later, random walks of this type converge almost surely to the boundary. Equivalently, one can also use the Poincaré disc model. The disc has a natural topological boundary, i.e. the circle.

1.1 Questions

1. Does a typical sample path escape to ∞ or it comes back to the origin infinitely often?

   A random walk is *recurrent* if for any x, the probability that $g_n = x$ infinitely often is 1:

   $$P(g_n = x \text{ i.o.}) = 1$$

   Otherwise it is said to be *transient*.
2. If it escapes, does it escape with "positive speed"?

   $$l = \lim_{n\to\infty} \frac{d(g_n x, x)}{n}$$

   (if it exists)
3. Does a sample path track geodesics in X? How closely?

4. If X has a topological boundary ∂X, does a typical sample path converge to ∂X?
   If so, define the *hitting measure* ν on ∂X as
   $$\nu(A) = P(\lim_{n\to\infty} g_n x \in A)$$
   for any $A \subset \partial X$.
5. What are the properties of hitting measure? Is it the same as the geometric measure? For example, is the same as the Lebesgue measure?
6. What is the boundary of a group?

1.2 Simple random walks on abelian groups

Theorem 1.1 (Polya)**.** The simple random walk on $\mathbb{Z}^d$ is recurrent iff $d = 1, 2$.

Proof. $d = 1$:
Define $p^k(x, y) = \mathbb{P}(g_n = y \text{ if } g_0 = x)$. Look at $p^{2n}(0, 0)$. □

Lemma 1.1. $m = \sum_{n\geq 1} p^n(0, 0)$, then RW is recurrent iff $m = \infty$.

m is the "average number of visits to 0" .

Proof. $u := \mathbb{P}(\text{ return to 0 eventually })$
$\mathbb{P}(\# \text{ visits } \geq k) = u^k$

$$\begin{aligned}\mathbb{E}(\# \text{ of visits }) &= \sum_{k\geq 0} \mathbb{P}(\# \text{ visits } \geq k) \\ &= \sum_{k\geq 0} u^k = \frac{1}{1-u}\end{aligned}$$

If $u \neq 1$ then $\mathbb{E}(\ \#\text{ of visits })$ is finite .
∴ RW is recurrent iff $u = 1$ iff $m = \infty$. □

Now back to $p^{2n}(0, 0)$ ...
$p^{2n}(0, 0) = \frac{1}{2^{2n}}\binom{2n}{n}$ (choose $2n$ ways to go right)
Is $\sum_{n\geq 1} \frac{1}{2^{2n}}\binom{2n}{n}$ convergent?

Apply Stirling's Formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\begin{aligned}\frac{1}{2^{2n}}\binom{2n}{n} &= \frac{1}{2^{2n}}\frac{\sqrt{n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)^2} \\ &= \frac{1}{2^{2n}}\cdot\frac{\sqrt{n}}{n}\cdot\frac{\left(\left(\frac{2n}{e}\right)^{2n}\right.}{\left(\frac{n}{e}\right)^{2n}} \\ &= \frac{1}{\sqrt{n}}\end{aligned}$$

$\therefore$ our RW is recurrent .

$d = 2$:

$p^{2n}(0,0) = ?$

Now we have 4 directions to choose from.

$\therefore p^{2n}(0,0) = \frac{1}{4^{2n}}\binom{2n}{n}^2 \sim \frac{1}{n}$ (math left & right and match up and down)

$\therefore$ our RW is recurrent

What about $d = 3$? Can we think of this as $3RW$'s? which are independent?

$p^{2n}(0,0) = \frac{1}{6^{2n}}\binom{2n}{n}$

Suppose:

k up

j east

$n - k - j$ out

$$\begin{aligned}\therefore p^{2n}(0,0) &= \frac{1}{6^{2n}}\binom{2n}{n}\sum_{0\le k+j\le n}\left(\frac{n!}{k!j!(n-k-j)!}\right)^2 \\ &\le \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{\left(\left(\frac{n}{3}\right)!\right)^3}\sum_{0\le k+j\le n}\frac{n!}{k!j!(n-k-j)!}\end{aligned}$$

(The second factor is the # of ways to colour n balls in 3 colours)

$$\begin{aligned}\therefore p^{2n}(0,0) &\le \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{\left(\left(\frac{n}{3}\right)!\right)^3}\cdot 3^n \\ &\approx \frac{1}{n^{\frac{3}{2}}}\end{aligned}$$

$\therefore$ as a corollary we have the $\sum_{n\geq 0} p^{(n)}(0,0) < \infty$.
It is also possible to show that $p^{2n}(0,0) \approx n^{-\frac{d}{2}}$.

Theorem 1.2. The simple random walk on a 4-valent tree is transient.

Sketch of proof
The group is $G = \mathbb{F}_2 =< a, b >$. We want to look at $d_n = d(g_n x, x)$. If you give the position of the n^{th} step, then finding d_{n+1} is as folllows:

$$[d_{n+1} = \begin{cases} d_n + 1 & \mathbb{P} = \frac{3}{4} \text{ if } d_n > 0 d_n + 1 if d_n = 0 \\ d_n - 1 & \mathbb{P} = \frac{1}{4} \end{cases}$$

$\therefore\ \mathbb{E}(d_{n+1} - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ and $\mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2}$
If we know that $\lim_{n\to\infty} \frac{d_n}{n}$ exists almost surely and is constant, then

$$\lim_{n\to\infty} \frac{d_n}{n} \geq \frac{1}{2} \text{ a.s. as } \mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2} \Rightarrow \text{RW is transient}$$

Definition 1.3. The rate of escape or drift or speed of a RW is

$$\ell = \lim_{n\to\infty} \frac{d_n(g_n x, x)}{n}$$

(if it exists)

Note that for a 4-valent tree, $\ell \geq \frac{1}{2}$ (it is actually $\ell = \frac{1}{2}$) .
For $\mathbb{Z}^d$ (think of d independent walks on each coordinate) and a balanced RW (e.g. a simple RW), $\ell = 0$.

2 An introduction to Ergodic Theory

Ergodic theorems are theorems that tell you some average exists. Famous ones include Birkhoff, Von Neumann, Kingman, and the martingale convergence theorem.

2.1 Framework

Let $(X, \mathbb{P}, \mathcal{A})$ be a probability space — i.e. X is a measure space, $\mathbb{P}$ is a probability measure, and $\mathcal{A}$ is a σ-field .

Definition 2.1. Let $T : X \to X$ be a measurable map (dynamics) . T is measure preserving if for any $A \in \mathcal{A}$

$$\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$$

Example 2.1. $T(x) = x + \alpha \mod 1, \quad X = \mathbb{R}/\mathbb{Z}, \quad \alpha \in \mathbb{R}, \quad \mathbb{P} = \text{Lebesgue}$

Example 2.2. $T(x) = 2x \mod 1, \quad X = \mathbb{R}/\mathbb{Z}, \quad \mathbb{P} = \text{Lebesgue}(\text{“}\textit{expanding}\text{''})$

1 & 2 preserve the same measure. It's easy to see (1) is measure preserving since it just translates. For (2), the measure can come from two different places, but adding these places together gives the same measure.

Definition 2.2. An observable is a meaurable function $f : X \to \mathbb{R}$. $(f(T^n x))$.

Definition 2.3. A system (a measurable dynamical system $(X, \mathbb{P}, \mathcal{A}, T)$) is *ergodic* if T-invariant sets have either 0 or 1 measure.

Definition 2.4. A is T-invariant if $T^{-1}A = A$.

Definition 2.5. f is T-invariant if $f \circ T = f$ a.e.

Lemma 2.1. Irrational rotations of $\mathbb{R}/\mathbb{Z}$ are ergodic (rational rotations are not ergodic) .

Proof. Let $f \in \mathcal{L}^2(\mathbb{R}/\mathbb{Z}, m)$ such that $f \circ T = f$
Write Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

Then

$$\begin{aligned} f(Tx) &= \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n (x+\alpha)} \\ &= \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} e^{2\pi i n \alpha} \qquad \text{b/c } T = x + \alpha \\ &\Rightarrow a_n = a_n e^{2\pi i n \alpha} \\ &\Rightarrow a_n = 0 \text{ or } e^{2\pi i n \alpha} = 1 \end{aligned}$$

If $\alpha \notin \mathbb{Q}$ then $a_n = 0$ for $n \neq 0$ and $f = a_0$. □

Lemma 2.2. $T : X \to X$ is ergodic iff all invariant functions in $L^2(X, \mathbb{P})$ are constant.

Proof.
($\Rightarrow$)
Suppose that T is ergodic and f is invariant, $f \circ T = f$. Let $A = \{x : f(x) \geq c\}$, $T^{-1}A = \{x : f(Tx) \geq c\}$ for some c. Then $\mathbb{P}(A) = 0$ or $1 \Rightarrow f$ is constant a.e.
($\Leftarrow$)
If $T^{-1}A = A$, take $f = \chi_A$ (ch. function) which is T-invariant.Then $\chi_A \circ T = \chi_{T^{-1}A} = \chi_A \Rightarrow \chi_A$ is constant a.e. $\Rightarrow \mathbb{P}(A) = 0$ or 1 .

□

Consider 2^n. What is the distribution of the 1^{st} (base 10) digit of 2^n?

Theorem 2.1 (Birkhoff Ergodic)**.** Let $f \in \mathcal{L}^1(X, \mathbb{P})$, T measure preserving. Then for a.e. $x \in X$, the limit

$$\overline{f}(x) = \frac{f(x) + f(Tx) + f(T^2x) \cdots + f(T^n x)}{n} \quad \text{exists.}$$

Moreover, $\overline{f}$ is T-invariant and $\|\overline{f}\| \leq \|f\|$. Moreover, the convergence is also in $\mathcal{L}^1$ and $\int \overline{f}(x)\ d\mathbb{P}(x) = \int f(x)\ d\mathbb{P}(x)$.

Corollary 2.1. If T is ergodic, then $\overline{f}$ is constant a.e.

Note that if T is measure preserving (mps), then for any observable f

$$\int f \circ T = \int f$$
$$\int f(Tx)\ d\mathbb{P}(x) = \int \ f(x) dT_{\#}\mathbb{P}(x)$$

(b/c T mps $\longrightarrow = \displaystyle\int f(x)\ d\mathbb{P}(x)$)

$T_{\#}\mathbb{P}$ is the push forward measure i.e. $T_{\#}\mathbb{P}(A) = \mathbb{P}(T^{-1}A)$

Example 2.3. Take a random $x \in [0,1]$ (for Lebesgue measure) . Look at its base-10 expansion . What is the frequency of digit 5?
Consider $T(x) = 10x(\text{mod } 1)s$ mps .
$f(x) = \chi_{[0.5,0,6)}$.

$$f \circ T^k(x) = \begin{cases} 0 \text{ if } k^{\text{th}} \text{ digit is not } 5 \\ 1 \text{ if } k^{\text{th}} \text{ digit is } 5 \end{cases}$$

Definition 2.6. $S_n f(x) := f(x) + \ldots + f(T^{n-1}x)$ is called the Birkhoff sum and $\frac{T_n f}{n}$ is called the Birkhoff average (or ergodic average) .

Theorem 2.2 (Von Neumann Ergodic). $T : X \to X$ measure preserving, $f \in \mathcal{L}^2(X, \mathbb{P})$. Then the limit

$$\frac{f(x) + f(Tx) + \cdots + f(T^{n-1}x)}{n}$$

exists in $\mathcal{L}^2$.

2.2 Random Walk Setup

Consider $GX, \mu \in P(G)$. Where $\Omega = G^{\mathbb{N}} = \{(X_1, X_2, \dots, X_n, \dots)\}$ and $\mathbb{P} = \mu^{\mathbb{N}}$. $(\Omega, \mathbb{P})$ is called the step space of the RW $g_n = X_1 \cdots X_n$. The space $\tilde{\Omega} = G^{\mathbb{N}} = \{(g_1, g_2, \dots, g_n, \dots)\}$ is called the sample space .

$$(\Omega, \mathbb{P}) \to \tilde{\Omega}$$
$$(X_1, \dots) \mapsto g_n = X_1 \cdots X_n$$
$$T : \Omega \to \Omega \text{ shift}$$
$$(X_1, \dots, X_n, \dots) \mapsto (X_2, X_3, \dots)$$

- T is measure preserving WRT $\mathbb{P} = \mu^{\mathbb{N}}$.
- $f : \Omega \to \mathbb{R}$, $f(X_1, X_2, \dots) = \|X_1\|$.
- $\therefore\ f \circ T^k(X_1, X_2, \dots) = \|X_{k+1}\|$
- $d_n((X_1, \dots, X_n, \dots)) = d(X_1, \dots, X_n x, x)$ (x = starting pt.) "distance of n^{th} step of RW" .
- A sub-additive co-cycle on $(X, \mathbb{P}, \mathcal{A}, T)$ is a function $a : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that $a(n+m, x) \le a(n, x) + a(m, T^n x) \quad \forall n, m \in \mathbb{N}, x \in X$

Theorem 2.3 (Kingman's Ergodic Theorem)**.** Suppose $a_n(x) \in \mathcal{L}^1(X, \mathbb{P})$ and $\forall n\ a_n(x) \ge 0$, then for a.e. $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} a(n, x) \text{ exists .}$$

3 The Subadditive Ergodic Theorem

Let $(\Omega, \mathbb{P})$ be a probability measure space. A *subadditive cocycle* is a function $a : \mathbb{N} \times \Omega \to \mathbb{R}$ such that $a(m+n, \omega) \le a(n, \omega) + a(m, T^n \omega)$ where $T : \Omega \to \Omega$ is a measure-preserving map.

Example 3.1. Let $T : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ be the shift operator; i.e. $T((X_1, X_2, \cdots)) = (X_2, X_3, \cdots)$. Then $d(n+m, \omega) = d(\omega, X_1 \cdot X_2 \dots X_{n+m} \omega)$. Furthermore, $d(n, \omega) + d(m, T^n \omega) = d(\omega, X_1 \cdot X_2 \dots X_n \omega) + d(\omega, X_n + 1 \cdot X_2 \dots X_{n+m} \omega)$.
Is it true that

$$d(\omega, X_1 \cdot X_2 \dots X_{n+m} \omega) \le d(\omega, X_1 \cdot X_2 \dots X_n \omega) + d(\omega, X_{n+1} \cdot X_{n+2} \dots X_{n+m} \omega) \ ?$$

We act by isometries to find that:

$$\begin{aligned} & d(\omega, X_1 \cdot X_2 \dots X_n\omega) + d(\omega, X_{n+1} \cdot X_{n+2} \dots X_{n+m}\omega) \\ = \; & d(\omega, X_1 \cdot X_2 \dots X_n\omega) + d(X_1 \cdot \dots \cdot X_n\omega, X_{n+1} \cdot X_1 \dots X_{n+m}\omega) \end{aligned}$$

So the above is true as it is just by the triangle inequality.

We say that a is integrable if $a(n,\cdot) \in \mathcal{L}^1(\Omega, \mathbb{P}) \ \forall n \in \mathbb{N}$. Assume

$$\inf_n \frac{1}{n} \int_\Omega a(n,\omega) d\mu(\omega) > -\infty$$

Theorem 3.1 (Kingman)**.** Under the assumptions above (i.e. that $(\Omega, \mathbb{P})$ a probability space, a a subadditive cocycle, T a measure-preserving map and a integrable with $\inf_n \frac{1}{n} \int_\Omega a(n,\omega) d\mu(\omega) > -\infty$) then, for almost every $\omega \in \Omega$ we have

$$\overline{a}(\omega) = \lim_{n\to\infty} \frac{a(n,\omega)}{n} \text{ exists.}$$

Moreover, the convergence is also in $\mathcal{L}^1$, $\overline{a}$ is T-invariant ($\overline{a} \circ T = \overline{a}$ a.e.)

$$\lim_{n\to\infty} \frac{1}{n} \int_A a(n,\omega) d\mu(\omega) = \int_A \overline{a}(\omega) d\mu(\omega)$$

for all T-invariant measurable A.

Recall: the additive case. Then

$$\begin{aligned} f(\omega) := a(1,\omega) \quad & \text{and} \\ a(n,\omega) = f(\omega) + f(T\omega) + \cdots f(T^{n-1}\omega) = S_n f \end{aligned}$$

$S_n f$ is called the *Birkhoff sum.* The Birkhoff Ergodic Theorem tells us that

$$\lim_{n\to\infty} \frac{S_n f}{n} \text{ exists } a.s.$$

Proof of Subadditive Ergodic Theorem

(A. Karlsson, "A proof of the subadditive ergodic theorem")

Lemma 3.1. Let (v_n) be a subadditive sequence of reals; i.e. $v_{n+m} \leq v_n + v_m$. Then

$$\lim_{n\to\infty} \frac{1}{n} v_n = \inf_{m>0} \frac{v_m}{m} \in \mathbb{R} \cup \{-\infty\}$$

Proof. : Take $\epsilon > 0$ and M such that

$$\frac{v_M}{M} \le \inf_m \frac{v_m}{m} + \epsilon$$

(which exists by the definition of an infimum). Each n can be written as $n = k_n M + r_n,\ 0 \le r_n < M$. Then

$$\begin{aligned}
\frac{k_n}{n} &\to \frac{1}{M} \text{ (because } r_n \text{ small, } M \text{ fixed)} \\
\implies \inf_m \frac{1}{m} v_m &\le \frac{1}{n} v_n \\
&= \frac{1}{n} v_{k_n M + r_n} \\
&\le \frac{1}{n}(k_n v_M + v_{r_n}) \\
&\le \frac{1}{M} v_M + \epsilon \le \inf_m \frac{v_m}{m} + 2\epsilon \\
\implies \lim_{n\to\infty} \frac{1}{n} v_n &= \inf_m \frac{v_m}{m}
\end{aligned}$$

□

Lemma 3.2 (Riesz)**.** Let $c_0, c_1, \cdots, c_{n-1}$ be a finite sequence of numbers. Call a term c_u a leader if one of $c_u, c_u + c_{u+1}, \cdots, c_u + c_{u+1} + \cdots + c_{n-1}$ is negative. Then, the sum of the leaders is non-positive.

Proof.

We prove this by induction. If $n = 1$ then the claim is trivial. Now, construct the sequence $c_0, c_1, \cdots, c_{n-1}$. If c_0 is not a leader then the claim follows by the inductive hypothesis, because we have that the leaders of $(c_0, c_1, \cdots, c_{n-1})$ are the same as the leaders of $(c_1, \cdots, c_{n-1})$, a shorter sequence.

Now, suppose c_0 is a leader. Then $\exists k$ smallest such that $c_0 + c_1 + \cdots + c_k < 0$. Therefore c_i for $i \le k$ are also leaders. But now since the sequence $(c_{k+1}, \cdots, c_{n-1})$ is a shorter sequence than the original, by strong induction the leaders in this sequence have a non-positive sum. Therefore, $c_0 + c_1 + c_k + (\text{leaders of}(c_{k+1}, \cdots, c_{n-1}))$ is non-positive.

□

Lemma 3.3. Let $a(n, \omega)$ be a subadditive cocycle. Then the functions

$$f(\omega) = \limsup \frac{1}{n} a(n, \omega)$$
$$g(\omega) = \liminf \frac{1}{n} a(n, \omega)$$

are a.e. T-invariant.

Proof.
By definition, we have that $a(n, T\omega) \geq a(n+1, \omega) - a(1, \omega)$. Therefore

$$\limsup \frac{1}{n} a(n, T\omega) \geq \limsup\left(\frac{a(n+1, \omega)}{n+1} - \frac{a(1, \omega)}{n}\right)$$
$$\implies f(T\omega) \geq f(\omega)$$

Similarly, $g(T\omega) \geq g(\omega)$. But now since T is measure preserving and μ is T-invariant,

$$\int f \circ T d\mu = \int f d\mu$$
$$\int g \circ T d\mu = \int g d\mu$$
$$\implies f \circ T = f, \quad g \circ T = g$$

□

Aside: $f(\omega)$ and $g(\omega)$ are integrable as

$$\int a(n, \omega) \leq \int (a(1, \omega) + \cdots + a(1, T^{n-1}\omega))$$
$$\implies \int a(n, \omega) \leq n \int a(1, \omega)$$
$$\implies \int \frac{a(n, \omega)}{n} \leq \int a(1, \omega)$$

where we know that $a(1, \omega)$ is integrable by assumption.

Proposition 3.1 (Maximal Ergodic Inequality)**.**
Let $a(n,\omega)$ be a subadditive cocycle. Let

$$B = \Big\{\omega : \liminf_n \frac{1}{n} a(n,\omega) < 0\Big\}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \int_B a(n,\omega) d\mu(\omega) \leq 0$$

Proof.
Note that $T^{-1}B = B$ up to measure zero (by the previous lemma). Let

$$A_n = \Big\{\omega : \inf_{1\leq k\leq n} a(k,\omega) < 0\Big\}$$

$$\Psi_n = \Big\{\omega : \inf_{1\leq k\leq n} [a(n,\omega) - a(n-k, T^k\omega)] < 0\Big\}$$

Then we have

$$\begin{aligned} &a(n,\omega) \leq a(k,\omega) + a(n-k, T^k\omega) \\ &\implies a(n,\omega) - a(n-k, T^k\omega) \leq a(k,\omega) \\ &\implies A_n \subset \Psi_n \end{aligned}$$

Also, $A_n \subset A_{n+1}$ and $B \subset \bigcup_n A_n$. Let $b_n(\omega) = a(n,\omega) - a(n-1, T\omega)$. Then

$$\begin{aligned} a(n,\omega) - a(n-k, T^k\omega) &= b_n(\omega) + b_{n-1}(T\omega) + \cdots + b_{n-k+1}(T^{k-1}\omega) \\ a(0,\omega) &= 0 \\ a(n,\omega) &= \sum_{0\leq k\leq n-1} b_{n-k}(T^k\omega) \\ &\implies T^k\omega \in \Psi_{n-k} \\ &\iff \exists j, k \leq j \leq n-1, \text{ such that } b_{n-k}(T^k\omega) + \cdots + b_{n-j}(T^j\omega) < 0 \end{aligned}$$

By the lemma on leaders, we have that

$$\sum_{0\leq k\leq n-1\ ;\ T^k\omega\in\Psi_{n-k}} b_{n-k}(T^k\omega) \leq 0$$

Therefore,

$$
\begin{aligned}
0 &\geq \sum_{0\leq k\leq n-1} \int_{B\cap T^-k\Psi_{n-k}} b_{n-k}(T^k\omega)d\mu(\omega) \\
&= \sum_{0\leq k\leq n-1} \int_{B\cap\Psi_{n-k}} b_{n-k}(\omega)d\mu(\omega) \\
&= \sum_{i=1}^{n} \int_{B\cap\Psi_i} b_i(\omega)d\mu(\omega)
\end{aligned}
$$

$$
\frac{1}{n}\int_B a(n,\omega)d\mu(\omega) = \frac{1}{n}\sum_{j=1}^{k}\int_{B\cap\Psi_i} b_i(\omega)d\mu(\omega) + \frac{1}{n}\sum_{i=1}^{n}\int_{B\setminus\Psi_i} b_i(\omega)d\mu(\omega)
$$

But $A_i \subset \Psi_i$ so $B \setminus \Psi_i \subset B \setminus A_i$, whence

$$
\begin{aligned}
b_i(\omega) &\leq a(1,\omega) \leq a^+(1,\omega) = \max\{0, a(1,\omega)\} \\
B &\subset \bigcup_{i=1}^{\infty} A_i \\
&\implies \frac{1}{n}\int_B a(n,\omega)d\mu(\omega) \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\int_{B\setminus\Psi_i} b_i(\omega)d\mu(\omega) \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\int_{B\setminus A_i} a^+(1,\omega)d\mu(\omega) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}
$$

□

4 Proof of the Subadditive Ergodic Theorem

Theorem 4.1 (Kingman)**.** If $a : \mathbb{N}\times\Omega \to \mathbb{R}$ is a subadditive cocycle on a measurable probability space $T : (\Omega, \mu, \mathcal{M})$ such that $a(n,\cdot) \in \mathcal{L}^1(\Omega, \mathbb{R})\forall n$

$$
\inf_n \frac{1}{n}\int a(n,\omega)\ d\mu(\omega) > -\infty
$$

then a.s.

$$
\overline{a}(\omega) : \lim_{n\to\infty}\frac{1}{n}a(n,\omega)
$$

exists

Lemma 4.1 (Maximal Inequality)**.** If $b = \{\omega : \liminf_{n\to\infty} a(n,\omega) < 0\}$ then

$$\lim_{n\to\infty} \frac{1}{n} \int_B a(n,\omega) \le 0$$

Corollary 4.1. If $\widetilde{B} = \{\omega : \liminf \frac{1}{n} a(n,\omega) < \lambda\}$ then

$$\lim_{n\to\infty} \frac{1}{n} \int_{\widetilde{B}} a(n,\omega)\ d\mu \le \lambda \cdot \mu(\widetilde{B})$$

Proof.
Apply the lemma to $\widetilde{a}(n,\omega) = a(n,\omega) - n\lambda$ □

4.1 Case 1

If a is additive (Birkhoff), for $\alpha < \beta$

$$E_{\alpha,\beta} := \left\{\omega : \liminf \frac{1}{n} a(n,\omega) \le \alpha < \beta \le \limsup \frac{1}{n} a(n,\omega)\right\}$$

WTS $\mu(E_{\alpha,\beta}) = 0\ \forall \alpha, \beta$.
Note that $E_{\alpha,\beta}$ is T invariant (showed last time) . If $\mu(E_{\alpha,\beta}) > 0$, then consider a as a cocycle over $T|_{E_{\alpha,\beta}} : E_{\alpha,\beta} \to E_{\alpha,\beta}$.
Let $E = \left\{\omega : \liminf \frac{1}{n} a(n,\omega) < \alpha\right\} \supset E_{\alpha,\beta}$.
By the corollary we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{E_{\alpha,\beta}} a(n,\omega)\ d\mu(\omega) \le \alpha \cdot \mu(E_{\alpha,\beta})$$

Furthermore, we have that

$$\lim_{n\to\infty} \int_{E_{\alpha,\beta}} a(1,\omega)\ dmu = \lim_{n\to\infty} \frac{1}{n} \int_{E_{\alpha,\beta}} a(n,\omega)\ d\mu(\omega)$$

Because

$$\begin{aligned}\int a(n,\omega) &= \int a(1,\omega) + \int a(n-1,\omega) \\ &= \cdots = n \cdot \int a(1,\omega)\end{aligned}$$

$$\therefore \lim_{n\to\infty} \int_{E_{\alpha,\beta}} a(1.\omega)\ d\mu = \lim_{n\to\infty} \frac{1}{n} \int_{E_{\alpha,\beta}} a(n,\omega)\ d\mu$$
$$\le \alpha \cdot \mu(E_{\alpha,\beta})$$

Because a is additive $\implies$ $-a$ is additive So by a similar argument as above:

$$-\int_{E_{\alpha,\beta}} a(1,\omega)\ d\mu(\omega) \le -\beta \cdot \mu(E_{\alpha,\beta})$$

$\therefore\ \beta \cdot \mu(E_{\alpha,\beta} = 0$

4.2 Case 2 – *the general sub-additive case*

Consider $v_n(\omega) = a(n,\omega) - \sum_{i=0}^{n-1} a(1,T^i\omega) \le 0$.
Let $\widetilde{a}(n,\omega) = \sum_{i=0}^{n-1} a(1,T^i\omega) \le 0$ (a Birkhoff sum) .
$\therefore\ \widetilde{a}(n,\omega)$ is additive $\implies$ apply case 1
We need to show

$$\lim_{n\to\infty} \frac{1}{n} v_n(\omega)$$

exists.
Let

$$g(\omega) = \liminf_n \frac{1}{n} v_n(\omega)$$
$$f(\omega) = \limsup_n \frac{1}{n} v_n(\omega)$$

For any $\alpha > 0$, let

$$B = \{\omega : f(\omega) - g(\omega) > \alpha\}$$
$$\mu(B) = 0$$

By the first lemma

$$\begin{aligned}\gamma(v) &= \lim_{n\to\infty} \frac{1}{n} \int v(n,\omega)\ d\mu(\omega)\\ &= \inf_n \frac{1}{n} \int v(n,\omega)\ d\mu(\omega)\end{aligned}$$

Begin an $\epsilon > 0$ and pick M such that $m > M$

$$\begin{aligned}\frac{1}{m} \int v_n &\le \gamma(v) + \epsilon\\ g^M(\omega) &= \liminf_n \frac{1}{nM} v_{nM}(\omega)\\ f^M(\omega) &= \limsup_n \frac{1}{nM} v_{nM}(\omega)\\ \therefore\ g^M(\omega) &\ge g(\omega) \quad \text{and} \quad f^M(\omega) \le f(\omega)\end{aligned}$$

Claim:

$$f^M(\omega) = f(\omega),\ g^M(\omega) = g(\omega)$$

For $0 \le k < M$:

$$\begin{aligned}v_{(n+1)M}(\omega) &\le v_{nM+k}(\omega) + v_{M-k}(T^{nM+k}\omega)\\ &\le v_{nM+k}(\omega)\\ &\le v_{nM}(\omega)\\ \therefore\ f^M(\omega) &= \limsup_n \frac{v_{(n+1)M}}{(n+1)M}\\ &\le \limsup_{n,k} \frac{v_{nM+k}}{nM+k}\\ &\le \limsup_n \frac{v_n M}{nM}\\ &= f(\omega)\end{aligned}$$

Similarly, $g^M(\omega) = g(\omega)$.
Therefore, let

$$f_n^M(\omega) = \underbrace{v_{nM}(\omega)}_{\le 0} - \sum_{i=0}^{n-1} v_M(T^{iM}\omega) \le 0 \qquad (\star)$$

$$
\begin{aligned}
f - g = f^M - g^M &= \limsup \frac{1}{nM} v_n^M - \liminf \frac{1}{nM} v_{nM} \\
&= \limsup \frac{1}{nM} v_n^M - \liminf \frac{1}{nM} v_n^M \\
&\leq - \liminf \frac{1}{nM} v_n^M
\end{aligned}
$$

Let $B = \{\omega : f(\omega) - g(\omega) > \alpha\} \subset \left\{\omega : \liminf \frac{1}{n} v_n^M < -M_\alpha \right\}$

$(\star)$ tells us that

$0 \geq \gamma(v^M) = M\gamma(v) - \int v_m$

and the way we chose M tells us that

$$
M\gamma(v) - \int v_M \geq -M\epsilon
$$

By the maximal inequality (last class)

$$
\begin{aligned}
-M\alpha\mu(E) &\geq \lim \frac{1}{n} \int_E v_n^M \\
&\geq \lim \frac{1}{n} \int_\Omega v_n^M \\
&\geq -M\epsilon \\
&\implies \mu(E) \leq \frac{\epsilon}{\alpha} \to 0 \text{ as } \epsilon \to 0 \\
&\implies \liminf \frac{1}{n} a(m,\omega) = \limsup \frac{1}{n} a(n,\omega) \text{ a.s.}
\end{aligned}
$$

$\overline{a}(\omega) = \lim_n \frac{1}{n} a(n,\omega)$ is T−invariant. If T is ergodic then only invariant functions are constant and $\overline{a} = A$ a.s. (constant)

Application

$$
\begin{aligned}
&\Omega = G^{\mathbb{N}} \\
&\mathbb{P} = \mu^{\mathbb{N}} \\
&T = \text{ shift is ergodic}
\end{aligned}
$$

Proposition 4.1. Let (G, μ) be a random walk on G. Let G act as (X, d) by isometries.

Definition 4.1. The measure μ has finite first moment if $\int a\, d(x, gx) d\, u(g) < \infty$ for some $x \in X$. What this means: a finite first moment means that the average first step is bounded i.e. the probability that in the first step you go an infinite distance has probability zero

Example 4.1. $G, \mu = \delta_{g_1} + \cdots + \delta_{g_r}$ has finite first moment (as it is finitely supported)

$$d(x, gx) \le \max_{1 \le i \le r} d(x, g_i(x)) < \infty$$

because in the metric spaces we're using, every 2 points have finite distance

Example 4.2. $G = \mathbb{Z}, X = \mathbb{R}, \mu(k) = \dfrac{c_1}{k^2}$

$$\therefore \int_G d(gx, x) d\mu(g) = \sum_{k \in \mathbb{Z}} |k| \frac{c_1}{k^2} = \infty$$

$$\therefore \text{ no finite first moment}$$

But if $\mu(k) = \dfrac{c_2}{|k|^3}$ then:

$$\int_G d(gx, x) d\mu(g) = \sum_{k \in \mathbb{Z}} |k| \frac{c_2}{|k|^3} < \infty$$

Exercise 4.1. If $x, y \in X$ and $\int d(x, gx) d\mu(y) < \infty$ prove that $\int d(y, gy) d\mu(y) < \infty$

Proof.

$$\begin{aligned} \int d(y, gy) &\le d(y, x) + d(x, gx) + d(gx, gy) \\ &\le 2d(x, y) + d(x, gx) \end{aligned}$$

$\therefore \int d(y, gy) \le 2d(x, y) - \int d(x, gx) d\mu(g)$ □

If μ has finite first moment then there exists $A \ge 0$ such that a.s.

$$\lim_{n \to \infty} \frac{d(g_n x, x)}{n} = A$$

We call A the drift or rate of escape of a RW.

Proof. (By Kingmans Theorem)
Let $\Omega = G^{\mathbb{N}}$. Let $a(n,\omega) = d(x, g_n x)$.
You need to check that $a(n,\cdot) \in \mathcal{L}^1(\Omega,\mathbb{P}) \forall n \geq 0$.
Finite first moment iff $a(1,\cdot) \in \mathcal{L}^1$.
So this falls out immediately because

$$\begin{aligned}\int a(n,\omega) &\leq \int a(1,\omega) + \int a(n-1,T\omega)\\ &\leq a(1,\omega) + \int a(n-1,T\omega)\\ &\leq n\int a(1,\omega) < \infty\end{aligned}$$

□

Example 4.3. T mps on $(X,\mathbb{P},\mathscr{F})$
$\mathscr{F}^T = \{A \in \mathscr{F} : T^{-1}A = A \text{ up to measure } 0\}$
$\mathscr{F}^T \subset \mathscr{F}$
$f : X \to \mathbb{R}$
f is $\mathscr{F}^T-$ measurable iff $f \circ T = f$ a.s.

5 Martingales

5.1 Conditional Expectations

Let

- $(X,\mathcal{A},\mu)$
- $f \in \mathcal{L}^1(X,\mathcal{A},\mu)$
- $\mathcal{B} \subset \mathcal{A}$
- $\mathcal{L}^1(X,\mathcal{B},\mu) \subset \mathcal{L}^1(X,\mathcal{A},\mu)$

Define the conditional expectation of f w.r.t. $\mathcal{B}$ to be a function $g \in \mathcal{L}^1(X,\mathcal{B},\mu)$ which is $\mathcal{B}$–measurable and s.t. $\int_B f\ d\mu = \int_B g\ d\mu$ for $B \in \mathcal{B}$.
Denote

$$g = \mathbb{E}[f|\mathcal{B}].$$

Example 5.1. $T : X \to X$ μ−preserving measurable w.r.t $\mathcal{A}$.
$\mathcal{B} = \mathcal{A}^T = \{A \in \mathcal{A} : T^{-1}A = A \text{ up to measure } 0\}$.
f is a $\mathcal{A}^T$− measurable $\iff$ $f \circ T = f$ a.s. $\{x : f(x) \geq c\}$ is T inv. for every c .

Theorem 5.1 (Birkhoff)**.** If $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ then $\bar{f}(x) = \lim_n \frac{f(x) + f(Tx) + \cdots + f(T^{n-1}x)}{n}$ exists in $\mathcal{L}^1$ a.s. and is T−invariant .

Claim

$$\mathbb{E}[f|\mathcal{A}^T] = \bar{f}$$

Proof. If $B \in \mathcal{A}^T$

$$\int_B f d\mu = \int_B \bar{f}\, d\mu = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_B f \circ T^k d\mu$$

Invariance of $\mathcal{B}$ implies that

$$\int_B f \circ T^k d\mu = \int_{T^{-k}B} f \circ T^k d\mu$$

Then

$$\int_B f d\mu = \int \chi_B f d\mu = \int \chi_B \circ T^k f \circ T^k d\mu$$

□

Remark
If $f \in \mathcal{L}^2(X, \mathcal{A}, \mu) = H$ (Hilbert space) then $H_1 = \mathcal{L}^2(X, \mathcal{B}, \mu) \subset H$ is a closed subspace and $\mathbb{E}[f|\mathcal{B}]$ is the orthogonal projection of f onto H_1 .
In general, if $H_1 \subset H$ is a closed linear subspace of a Hilbert space, then there is a projection operator $P : H \to H$ s.t. $\text{im}P = H_1$ and $P^2 = P$.

Example 5.2. $(X_m) : \Omega \to \mathbb{R}$ a sequence of random variables
$\mathcal{F} = \sigma(X_1, \dots, X_n)$

Definition 5.1. $\sigma(X_1, \dots, X_k)$ is the smallest σ− algebra Σ s.t. all X_i are Σ−measurable

If $f : \Omega \to \mathbb{R}$ then $\mathbb{E}[f|\mathcal{F}]$ is the expectation of f given you know the values $X_1, \dots, X_n$.

Example 5.3.

- X_n i.i.d.
- $\mathbb{P}(X_n = 1) = \dfrac{1}{2} = \mathbb{P}(X_n = 0)$
- $\sigma(X_1, \ldots, X_n)$
- $\Omega = \{0,1\}^{\mathbb{N}}$
- $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is $\sigma(X_1, \ldots, X_n)$-measurable
- f is $\sigma(X_1)$-measurable

$$\begin{cases} \varnothing \\ \varnothing \times \{0,1\}^{\mathbb{N}} \\ \{1\} \times \{0,1\}^{\mathbb{N}} \\ \Omega \end{cases}$$

$\sigma(X_1, \ldots, X_n) = \left\{ A \times \{0,1\}^{\mathbb{N}}, A \subset \{0,1\}^{\mathbb{N}} \underline{\text{ any subset }} \right\}$

What does it mean for $f : \Omega \to \mathbb{R}$ to be measurable for $\sigma(X_1, \ldots, X_n)$?
The function is <u>determined</u> by the first n coordinates

$$f(x_1, \ldots, x_n, \ldots) = f(x_1, \ldots, x_n)$$

Example 5.4. Consider $S_n = X_1 + \cdots + X_n$
$\mathbb{E}[S_{100}|X_1, X_2, X_3] \stackrel{?}{=}$
$\mathbb{E}[S_3|X_1, X_2, X_3] = X_1 + X_2 + X_3$
If $X_1 = X_3 = 1$ and $X_2 = 0$ then

$$\mathbb{E}[S_4|X_1, X_2, X_3] = 2.5$$

As a function,

$$\mathbb{E}[S_4|X_1, X_2, X_3] = X_1 + X_2 + X_3 + \frac{1}{2}$$

Therefore,

$$\mathbb{E}[S_{100}|X_1, X_2, X_3] = X_1 + X_2 + X_3 + \frac{97}{2}$$

5.2 Properties

Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) : X_1, \ldots, X_n, \mathcal{F} = \sigma(X_1, \ldots, X_n)$.
- If f and $X_1, \ldots, X_n$ are independent then $\mathbb{E}[f|\mathcal{F}]$...
E.g.
$\mathbb{E}[X_{50}|X_1, X_2, X_3] = \frac{1}{2}$ (X_{50} is independent of X_1, X_2, X_3)
Therefore, if f and $X_1, \ldots, X_n$ are independent then $\mathbb{E}[f|\mathcal{F}] = f$.
- If f is $\mathcal{F}-$ measurable then $\mathbb{E}[f|\mathcal{F}] = f$
- $\mathbb{E}[f + g|\mathcal{F}] = \mathbb{E}[f|\mathcal{F}] + \mathbb{E}[g|\mathcal{F}]$
Note that $\mathbb{E}[S_{n+1}|S_1, \ldots, S_n] = S_n + \frac{1}{2}$

Proof.
$S_{n+1} = S_n + X_{n+1}$
Therefore,

$$\begin{aligned}\mathbb{E}[S_n + X_{n+1}|S_1, \ldots, S_n] &= \mathbb{E}[S_n|S_1, \ldots, S_n] + \mathbb{E}[X_{n+1}|S_1, \ldots, S_n] \\ &= S_n + \mathbb{E}[X_{n+1}] = S_n + \frac{1}{2}\end{aligned}$$

Therefore, Case: $X_n = 1$ w.p. $\frac{1}{2}$ and $X_n = -1$ w.p. $\frac{1}{2}$

□

5.3 Martingales

Let $(X_n) : \Omega \to \mathbb{R}$ a sequence of random variables. It is a martingale if

$$\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = X_n$$

It is a sub-martingale if

$$\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] \leq X_n$$

or a supermartingale if $\geq$

5.4 The Martingale Convergence Theorem

Theorem 5.2 (Doob)**.** Let $(X_n) \subset \mathcal{L}^1(\Omega, \mu)$ be a sub-martingale such that $\sup_n \mathbb{E}(X_n) < \infty$. Then there exists a random variable $X_\infty : \Omega \to \mathbb{R}$ s.t.

$$X_n \to X_\infty \text{ almost surely.}$$

Remark 5.1. Equivalently, works for supermartingales .

Remark 5.2. The convergence need not be in $\mathcal{L}^1$.

Example 5.5.

$$X_n = \begin{cases} 0 & \mathbb{P} < \frac{1}{2} \\ 2 & \mathbb{P} = \frac{1}{2} \end{cases}$$

Then the product $Y_n = X_1 \cdots X_n$ is a martingale. In fact

$$\mathbb{E}[Y_{n+1}|Y_1, \ldots, Y_n] = \mathbb{E}[Y_{n+1} = X_{n+1} + Y_n|Y_n] =$$

$$= 2 \cdot Y_n \cdot \mathbb{P}(X_{n+1} = 2) + 0 \cdot Y_n \cdot \mathbb{P}(X_{n+1} = 0) = Y_n$$

Note (by Fatou)

$$\liminf \int X_n \geq \int X_\infty$$

5.5 Transience on a radially symmetric tree

A radially symmetric tree of valence $(a_1, a_2, \ldots)$ is a tree where all vertices at distance n from the base point have exactly a_{n-1} children .

Theorem 5.3. A RW on a radially symmetric tree $(a_1, a_2, \ldots)$ is transient iff

$$\sum_{n \geq 1} \frac{1}{a_1 \cdot a_2 \cdots a_n} < \infty$$

Theorem 5.4 (Blackwell)**.** Consider Markov chain on $\mathbb{N}$ defined by transition probability

$$\begin{cases} q(0,0) = 1 \\ q(n, n+1) = p_n \\ q(n, n-1) = q_n = 1 - p_n \end{cases}$$

where $n \geq 1$.

Then this Markov chain will eventually hit 0 almost surely iff the equation $f(n) = q_n \cdot f(n-1) + p_n \cdot f(n+1)$ has no non-constant bounded solution.

d_n = distance of the nth step from the origin and

$p_n = \dfrac{a_n}{a_n + 1}$

$q_n = \dfrac{1}{a_n + 1}$

6 Stationary Measures

Definition 6.1. A metric space B is a $G-$*space* if there is an action of G on B by homeomorphisms, i.e. a homomorphism $\rho : G \to \text{Homeo}(B)$.

Definition 6.2. A measure ν on B is $\mu-$*stationary* if

$$\sum_{g \in G} \mu(g)\ g_* \nu = \nu$$

Recall that $(g * \nu)(A) = \nu(g^{-1}A)$
$\left(\int_G g_*\nu\ d\mu(g) = \nu\right)$

Definition 6.3. The pair (B, ν) is a $(G, \mu)-$space if B is a $G-$space and ν is a $\mu-$stationary probability measure on B.

Idea
If B is topological boundary and RW converges a.s. then the hitting measure is stationary
Hitting measure:

$$\nu(A) := \mathbb{P}\left(\lim_{n\to\infty} w_n x \in A\right)$$

Lemma 6.1. If B is a compact G-space then there exists at least one μ-stationary measure ν on B .

6.1 A few facts in functional analysis

If B is a compact metric space, we denote

$$\mathcal{C}(B) = \{f : B \to \mathbb{R} \text{ continuous }\}$$

This is a Banach space with norm $\|f\| := \sup_{x\in B} |f(x)|$.
Given a measure ν on B, consider the map $\Phi_\nu : \mathcal{C}(B) \to \mathbb{R}$ defined as

$$\Phi_\nu(f) := \int_B f d\nu.$$

This gives a linear functional on the space of continuous functions .
Note that Φ_ν is a bounded functional, as $|\Phi_\nu(f)| \le \|f\|$.
We define the *dual space* to $\mathcal{C}(B)$ as $\mathcal{C}(B)^* = \{\text{bounded linear functionals on } \mathcal{C}(B)\}$
.

Theorem 6.1 (Riesz-Markov-Kakutani)**.** Let B be a compact metric space.
$\mathcal{C}(B)^* = \{$ signed Borel measures on $B\}$
In part., given any bounded linear functional Φ on $\mathcal{C}(B)$, there exists a (unique) signed measure ν s.t. $\Phi(f) = \int_B f d\nu \quad \forall f \in \mathcal{C}(B)$

Theorem 6.2 (Alaoglu-Banach)**.** If V is a normed vector space, then the unit ball of V^* is compact in the weak-$\star$ topology.

Definition 6.4 (weak-$\star$ convergence)**.**
$V^* \supset (\varphi_n) \overset{*}{\to} \varphi \in V^*$ if $\varphi_n(v) \overset{n\to\infty}{\to} \varphi(v) \ \forall v \in V$.

Example 6.1. $(\nu_n) \overset{*}{\to} \nu \in P(B)$ if for every $f \in \mathcal{C}(B)$ $\int f d\nu_n \to \int f d\nu$

Proof Alaoglu-Banach.
For each $v \in V$, consider $D_v = \{x \in \mathbb{R} : ||x|| \le ||v||\}$.
$B^* \overset{\Psi}{\hookrightarrow} \Pi_{v\in V} D_v$ compact (by Tychonoff) .
$\left(||\varphi|| = \sup_{v\in V} \frac{||\varphi(v)||}{||v||} \le 1 \right)$
$B^* = \{\varphi \in V^* : ||\varphi|| \le 1\}$
$B^* \ni \varphi \mapsto (\varphi(v))_{v\in V}$
Since $\varphi \in B^*$,
$|\varphi(v)| \le ||v|| \Rightarrow \Psi(B^*) \subset \Pi_{v\in V} D_v$
Need to prove $\Psi(B^*)$ is closed, hence compact.
$\varphi_n(v + w) = \varphi_n(v) + \varphi_n(w)$
$\varphi_\infty(v + w) = \varphi_\infty + \varphi_\infty(w)$

□

Theorem 6.3. The space of probability measure $P(B)$ on a compact metric space B is compact.

Proof. $P(B) \subset \{$ (unit ball for) signed measures$\} = \mathcal{C}(B)^*$
$$|\mu| = \sup_{f\in\mathcal{C}(B)\setminus\{0\}} \frac{||\mu(f)||}{||f||} = \sup \frac{|\int f d\nu|}{||f||} \le \sup \frac{||f||}{||f||} = 1$$
$P(B) = \{$ measures on B s.t. $|\mu| \le 1, \mu(1) = 1, \mu \ge 0\}$
where $\mu \ge 0$ means $\mu(f) \ge 0$ for all $f \ge 0$ since the conditions are closed, $P(B)$ is closed in a compact space, hence compact . □

Remark 6.1. It turns out that if B is a metric space, then $P(B)$ is compact $\iff$ B compact.

Idea
$\nu_n = P_\mu \circ P_\mu \circ \cdots \circ P_\mu(\nu)$

7 Boundary Theory

Let G be a group and μ a measure. Let G =Isom$^+(\mathbb{D})$ and $B = S^1$.

Definition 7.1. A metric space B is a G-space if G acts on B by homeomorphisms. A pair (B,ν) where $\nu \in P(B)$ is a (G,μ)-space if B is a G-space and ν is μ-stationary.

$$\sum_{g\in G} \mu(g) g * \nu = \nu$$
$$\int_G g * \nu \; d\mu(g) = \nu$$

Remark 7.1. There is a convolution operator

$$* : P(G) \times P(B) \to P(B)$$
$$(\mu,\nu) \mapsto \mu * \nu$$
$$\mu * \nu(A) = \int_G g * \nu(A) d\mu(g)$$

Consider action

$$\Phi : G \times B \to B$$
$$(g,x) \mapsto gx$$

Then $\Phi_*(\mu \times \nu) = \mu \star \nu$

Remark 7.2. A μ-stationary ν is a fixed point of operator

$$P_\mu : P(B) \to P(B). P_\mu(\nu) = \mu \star \nu$$

Lemma 7.1. If B is a compact G-space and $\mu \in P(G)$, then there exists at least a μ-stationary measure on B.

Proof.
$P_\mu : P(B) \to P(B)$ is a linear operator. $P(B)$ is compact . It has a fixed point. Let ν be any $\in P(B)$.
Consider
$$\nu_n = (P_\mu)^n(\nu)$$

Wrong proof

Then there exists a limit pt $\nu_\infty = \lim_k \nu_{n_k}$.

ν_∞ is a fixed pt: $P_\mu(\nu_\infty) = \lim_k P_\mu(\nu_{n_k}) = \lim_k P_\mu^{n_k+1}(\nu)$.

$\lim_k P_\mu^{n_k}(\nu) \stackrel{?}{=} \lim_k P_\mu^{n_k+1}(\nu)$.

$|\nu_{n_k} - \nu_{n_k+1}| \stackrel{?}{\to} 0$.

Correct proof

$\widetilde{\nu}_n = \frac{1}{n}(\nu_0 + \cdots + \nu_{n-1})$

By compactness, there is $\widetilde{\nu}_\infty = \lim_k \widetilde{\nu}_{n_k}$. Then $\widetilde{\nu}_\infty$ is μ-stationary .

$$P_\mu(\widetilde{\nu}_\infty) = \lim_{k\to\infty} P_\mu(\widetilde{\nu}_{n_k}) = \lim_{k\to\infty} \frac{1}{n_k}(\nu_1 + \cdots + \nu_{n_k})$$

$$\widetilde{\nu}_\infty = \lim_{k\to\infty} \widetilde{\nu}_{n_k} = \lim_{k\to\infty} \frac{1}{n_k}(\nu_0 + \cdots + \nu_{n_k-1})$$

$$\|P_\mu(\widetilde{\nu}_k) - \widetilde{\nu}_{n_k}\| = \frac{1}{n_k}\|\nu_{n_k} - \nu_0\| \le \frac{2}{n_k} \to 0$$

□

Theorem 7.1 (Schauder-Tychonoff)**.** Let $P : V \to V$ a bdd linear operator on a Banach space. If $B \subset V$ is a complex, compact subset, with $P(B) \subset B$ then there exists $v \in B$ s.t. $P(v) = v$.

Question: Is the stationary measure unique?

Problem: classify all stationary measures.

(We will see that on the Poincare disk they are unique)

Is every stationary measure the hitting measure of a RW?

Remark 7.3. Let $G \to G$ (X,d) by isometries and suppose action of G extends to an action by homeomorphisms to a "boundary", i.e. there exists action $G \to G$ $(X \cup \partial X)$.

G acts by homeos on ∂X and X is dense in $X \cup \partial X$.

Let $x \in X$. If for a.e. sample path, the limit

$$\lim_{n\to\infty} g_n x \text{ exists in } \partial X$$

We define the hitting measure ν on ∂X as

$$\nu(A) = \mathbb{P}\left(\lim_{n\to\infty} g_n x \in A\right)$$

Lemma 7.2. The hitting measure is μ-stationary (if it exists) .

Proof.

$$\begin{aligned} g_*\nu(A) &= \mathbb{P}\left(\lim_n g_n x \in g^{-1}A\right) \\ &= \mathbb{P}\left(\lim_n g g_n x \in A\right) \\ \sum \mu(g) g_*\nu(A) &= \sum \mu(g)\mathbb{P}\left(\lim_n g g_n x \in A\right) \\ &= \mathbb{P}\left(\lim_n g_n x \in A\right) = \nu(A) \end{aligned}$$

□

Proposition 7.1. Let (B,ν) be a (G,μ)-space .
Then for a.e. sample path (g_n) the limit $\lim_n (g_n)_*\nu$ exists in $P(B)$.
If $(g_n) = \omega \in \Omega$ is a sample path, we call limit $\nu_\omega = \lim_n (g_n)_*\nu$.

Lemma 7.3. If $f \in \mathcal{C}_0(B)$, then

$$X_n = \int_B f \; d((g_n)_*\nu) = \int_B f \circ g_n \; d\nu$$

is a martingale.

Proof.
$\mathbb{E}[X_{n+1}|X_n] \stackrel{?}{=} X_n$

$$\begin{aligned} \mathbb{E}\left[\int f(g_1\cdots \underbrace{g_{n+1}x}_{y = g_{n+1}x}) \; d\nu(x)|g_1\cdots g_n\right] &= \int_G\int_B f(g_1\cdots g_n g_{n+1}x) \; d\nu(x) \; d\mu(g_{n+1}) \\ &= \int_B f(g_1\cdots g_n y) \; d(\mu * \nu)(y) \\ &= \int_B f(g_1\cdots g_n y) \; d\nu(y) \end{aligned}$$

By the Martingale Convergence Theorem, for each $f \in \mathcal{C}^0(B)$ there is a.s. the limit

$$L(\omega, f) = \lim_{n\to\infty} \int f \; d((g_n) * \nu)$$

Pick sequence $(f_n) \subset \mathcal{C}^0(B)$, then there exists $A \subset \Omega$ of full measure s.t. $L(\omega, f_n)$ exists for all n, all $\omega \in A$. Pick (f_n) countable dense subset of $\mathcal{C}^0(B)$. Then there exists a continuous extension $\Phi : \mathcal{C}^0(B) \to \mathbb{R}$ which is a bdd linear functional.

Hence, there exists a measure $\nu_\omega \in P(B)$

$$\nu_\omega = \lim_{n\to\infty} \int (g_n)_* \nu$$

$$\forall f \in \mathcal{C}^0(B)$$

$$\int_B f \; d\nu_\omega = \lim_{n\to\infty} \int_B f \; d((g_n)_* \nu)$$

□

Fact

If B is compact metric, then $\mathcal{C}^0(B)$ is separable, i.e. there exists a countable dense subset.

7.1 Boundary Convergence

Theorem 7.2 (Furstenberg)**.** Let μ be a non-elementary probability measure on $SL_2(\mathbb{R})$ and pick $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. Then for almost every sample path (g_n) the limit $\lim_{n\to\infty} g_n(z)$ exists and belongs to $\partial\mathbb{H}$. A measure μ is non-elementary if its support is not contained in a compact subgroup of G and it does not preserve a finite set of points on ∂X .

$$(G : SL_2\mathbb{R}, X = \mathbb{H})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

$$\partial H = \mathbb{R} \cup \{\infty\}$$

7.2 Compact Subgroups

$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$

and its conjugates (gKg^{-1})

Example 7.1 (Parabolic subgroup)**.**

$P = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

$z \mapsto z + a$ in $\mathbb{H}$

P fixes 1 pt of ∂H

Example 7.2. Hyperbolic 1-generated subgroup

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^{+} \right\}$

$z \mapsto a^2 z$

H fixes the geodesic in $\mathbb{R}$.

H fixes 2 pts on ∂H .

Claim: every elementary subgroup of $SL_2\mathbb{R}$ is contained into either K, P, H or one of their conjugates. If a subgroup fixes a set $S \subset \partial\mathbb{H}$ with $|S| \geq 3$, then either it's trivial or it fixes a set S' with $|S'| \leq 2$.

8 Boundary Convergence

$$G = SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$$
$$X = \mathbb{H}^2$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az + b}{cz + d}$$

$\mathbb{H}^2$ has hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

with constant negative curvature (=-1) .

Geodesics

- Vertical lines
- Half circles orthogonal to real axis

Disc model

- $\mathbb{D}$
- geodesics = arcs of circle orthogonal to $\partial\mathbb{D}$

Theorem 8.1. (Furstenberg)

Let μ be a non-elementary measure on $SL_2(\mathbb{R})$ and consider a RW on $SL_2(\mathbb{R})$ with step distribution μ. Then for every $z \in \mathbb{H}^2$ for almost every sample path (g_n) the limit

$$\lim_{n\to\infty} g_n(z)$$

exists in $\partial\mathbb{H}^2$.

Proof.

$\partial\mathbb{H}^2$ is a compact metric space homeomorphic to S^1 and the action of $G = SL_2(\mathbb{R})$ on S^1 is by homeomorphisms. By previous lemma: there is a stationary measure ν on S^1 . Also: for almost every sample path (g_n),

$$\lim_{n\to\infty} (g_n)_*\nu = \nu_\omega \text{exists.}$$

Note that could also consider $B = \mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Recall:

Non-elementary means:

- supp μ not contained in compact subgroup
- supp μ does not fix a finite subset of $\partial\mathbb{H}^2$

□

Lemma 8.1. Let ν be a stationary measure on $B = \mathbb{H}^2 \cup \partial\mathbb{H}^2$. Then $\nu(\mathbb{H}^2) = 0$.

Remark 8.1. $\nu = \int_\Omega \nu_\omega \; d\mathbb{P}(\omega)$

Proof.

$$\int_\Omega (g_n)_*\nu \; d\mathbb{P}(\omega) = \int_G g_*\nu \; d\mu^n(g) \underbrace{=} \nu \text{ is } \mu^n \text{ stationary}\nu$$

$$\mu^n = \mu * \cdots * \mu(distribution of n^{\text{th}} step)$$

$$\int \nu_\omega \; d\mathbb{P}(\omega) = \int \lim_{n\to\infty} (g_n)_*\nu \; d\mathbb{P}(\omega) \underbrace{=}_{\text{dominated convergence}} \lim_{n\to\infty} \int \underbrace{(g_n)_*\nu}_{=\nu} \; d\mathbb{P}(\omega)$$

□

Proof of Lemma.

Assume supp μ is countable. Suppose $\nu(\mathbb{H}^2) = 1$. Then there exists compact $K \subset \mathbb{H}^2$ s.t. $\nu(K) > \frac{1}{2}$

$$\nu = \int \nu_\omega \; d\mathbb{P} = \int \lim_n (g_n)_* \nu \; d\mathbb{P}$$

Then there exists a positive measure set $A \subset \Omega$ s.t. $\nu_\omega(K) > \frac{1}{2}$ for $\omega \in A$. Hence there exists for $\omega \in A$ n_0 s.t. for $n \geq n_0$

$$(g_n)_* \nu(K) < \frac{1}{2} = \nu(g_n^{-1}(K))$$

Hence $K \cap g_n^{-1} K \neq \varnothing$. i.e. there is $y \in K$ s.t. $y = g_n^{-1} z, z \in K$.

$\square$

Recall

$\mathbb{H}^2 = \dfrac{SL_2(\mathbb{R})}{SO_2(\mathbb{R})}$

Pick $i \in \mathbb{H}^2$, $S = \text{Stab}(i) = \{g \in SL_2(\mathbb{R}) : g(i) = i\} = S)_2(\mathbb{R})$

$\widetilde{K} = KS \subset G$ (compact)

Then

$$\widetilde{y} = g_n^{-1} \widetilde{z} \Longrightarrow g_n^{-1} = \widetilde{y} \cdot \widetilde{z}^{-1}$$
$$g_n = \widetilde{z} \cdot \widetilde{y}^{-1} \in \widetilde{K} \widetilde{K}^{-1} = \underbrace{K_1}_{\text{compact}} \subset G$$

If $n + m \geq n \geq n_0$.

$g_{n+m} \in K_1, g_n \in K_1 \mapsto g_n^{-1} g_{n+m} \in K_1^{-1} K_1 = K_2$ (compact)

$g_n^{-1} g_{n+m} = (X_1 \cdots X_n)^{-1} (X_1 \cdots X_n \cdot X_{n+1} \cdots X_{n+m}) = X_{n+1} \cdots X_{n+m}$

Pick $(h_1, \ldots, h_m) \in (\text{supp } \mu)^m$.

By the ergodic theorem, for a.e. sample path ω there exists infinitely many n s.t.

$$\begin{cases} h_1 = X_{n+1} \\ h_2 = X_{n+2} \\ \vdots \\ h_n = X_{n+m} \end{cases}$$

(in particular, for a.e. $\omega \in A$)

Then

$$h_1 \cdots h_m \in K_2$$

So

$$\underbrace{< \text{supp } \mu >}_{\text{semigroup}} \subset K_2 \text{ (compact)}$$

Fact: a compact semigroup in a group is a group .

Exercise 8.1. A compact semigroup with cancellation is a group.
With cancellation means:

$$gh_1 = gh_2 \Rightarrow h_1 = h_2$$
$$h_1 g = h_2 g \Rightarrow h_1 = h_2$$

$\varphi_g : \Gamma \to \Gamma$ is injective
$\varphi_g(h) = gh$
Is it surjective if Γ is compact?
$\Gamma = \underbrace{< \text{supp } \mu >}_{\text{group}}$ (group) is closed inside compact set $\Rightarrow$ compact

Corollary 8.1. If ν is stationary on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, then $\nu(\partial\mathbb{H}^2) = 1$.

Lemma 8.2. ν is non-atomic .

Proof. Suppose ν is atomic, i.e. $\exists x \in \partial\mathbb{H}^2$ s.t. $\nu(\{x\}) > 0$.
$m = \sup_{x \in \partial\mathbb{H}^2} \nu(\{x\}) > 0$
$A_m = \{x \in \partial\mathbb{H}^2 : \nu(\{x\}) = m\}$
A_m is finite.
Let $x \in A_m$

$$\begin{aligned} \nu(\{x\}) &= \sum_g \mu(g) g_* \nu(\{x\}) \\ &= \sum_g \mu(g) \nu(\{g^{-1}x\}) \\ &\leq \sum_g \mu(g) \nu(\{x\}) = \nu(\{x\}) \end{aligned}$$

(recall that $\nu(\{x\})$ is of maximal weight)

Therefore you must have equality at all stages.

$$\sum \mu(g)\nu(\{g^{-1}x\}) = \sum \mu(g)\nu(\{x\})$$
$$\Rightarrow \text{ if } g \in \text{supp } \mu,\ \nu(\{g^{-1}x\}) = \nu(\{x\}) \Rightarrow g^{-1}x \in A_m$$

If $\Gamma = \underbrace{< \text{supp } \mu >}_{\text{semigroup}}$ then $\Gamma^{-1}A_m = A_m$. Then also A_m is invariant for the group generated by supp μ . But this contradicts non-elementary.

□

Lemma 8.3. Suppose $(g_n) \subset G = SL_2(\mathbb{R})$ and $x \in \mathbb{H}^2$ s.t. $g_n x \to \xi \in \partial\mathbb{H}^2$. Let ν be a non-atomic measure on $\partial\mathbb{H}^2$. Then there is a subsequence (n_k) s.t. $(g_{n_k})_*\nu \longrightarrow \delta_\xi$

Proof. Claim: There is a subsequence such that $g_{n_k}\eta \longrightarrow \xi$ for all $\eta \in \partial\mathbb{H}^2$ except for at most one point.
Suppose claim is <u>not true</u> .
Then by extracting a subsequence there are 2 points $\eta_1, \eta_2 \in \partial\mathbb{H}^2, b_1, b_2 \in \partial\mathbb{H}^2$ s.t.
$g_{n_k} \to \eta_1 \to b_1 \neq \xi$
$g_{n_k} \to \eta_2 \to b_2 \neq \xi$
Then $(g_n\eta_1, g_n\eta_2)$ cannot be a geodesic for n sufficiently large.

□

9 Boundary convergence II

Theorem 9.1 (Furstenberg)**.** Let μ be a non-elementary measure on $SL_2\mathbb{R}$. Then for any $z \in \mathbb{H}^2$ and for a.e. sample path (g_n) of RW with step distribution μ, the limit,

$$\lim_{n\to\infty} g_n(z) \in \partial\mathbb{H}^2 \text{ exists.}$$

Proof.

1. $\exists\nu$ a stationary measure on $\partial\mathbb{H}^2$
2. a.s. $\lim\limits_{n\to\infty}(g_n)_*\nu = \nu_\omega$ exists in $P(\partial\mathbb{H}^2)$
3. If for $x \in \mathbb{H}$, $g_n x \longrightarrow \xi \in \partial\mathbb{H}$, then there is a subsequence (n_k) s.t. $(g_{n_k})_*\nu \longrightarrow \delta_\xi$

□

Remark 9.1. If $g_n x \longrightarrow \xi \in \partial\mathbb{H}, x \in \mathbb{H}$ then $g_n y \longrightarrow \xi \in \partial\mathbb{H}$ for all $y \in \mathbb{H}$

Proof. $d(g_n x, g_n y) = d(x,y)$ is bounded independently of n.
In hyperbolic space, if $x_n \longrightarrow \xi \in \partial\mathbb{H}$ and $d(x_n, y_n) \leq \mathcal{C}$, then $y_n \longrightarrow \xi$
a.s. the sample path $(g_n x)$ is unbounded. (RW is invariant)
If there is positive probability that $g_n x \in B(x,R)$ for all n .
Then the *limit distribution* of RW assigns positive measure to $B(n,R)$.
But we saw that no stationary measure can charge $\mathbb{H}$ with positive measure.
$\nu_n = \mu^n * \delta_x \in P(\mathbb{H} \cup \partial\mathbb{H})$
$\widetilde{\nu}_n = \frac{1}{n}\sum_{k=0}^{n-1} \mu^k * \delta_x$
Take ν_∞ any limit of $(\widetilde{\nu}_n)$.
Then ν_∞ is stationary .
The sequence $(g_n x)$ has at least one limit point $\xi \in \partial\mathbb{H}$.
Thus, combining 2 and 3

$$\lim_{n\to\infty} (g_n)_* \nu = \delta_\xi$$

Hence, the sequence $(g_n x)$ has *exactly* one limit point on $\partial\mathbb{H}$.

□

We need to exclude the existence of limit points inside $\mathbb{H}$.

Proof.
If $g_{n_k} x \longrightarrow y \in \mathbb{H}$ then the sequence (g_{n_k}) is bounded in $SL_2(\mathbb{R})$.
Then there exists $C > 0$ s.t.

$$\frac{1}{C} \leq \frac{dg_{n_k}}{dx} \leq C$$

for all $x \in \partial\mathbb{H}$ and for all k.
Hence $\lim_{k\to\infty} (g_{n_k})_* \nu$ cannot be atomic.
$\left(\frac{1}{C}\nu(A) \leq g_* \nu(A) = \nu(g^{-1}A) \leq C\nu(A)\right)$

□

Remark 9.2. $\nu_n \xrightarrow{*} \nu$

It is not true that $\nu_n(A) \longrightarrow \nu(A)$ for any measurable A .
e.g.
$\nu_n = n^{\text{th}}$ step distribution of RW . ν = hitting measure

$$\begin{aligned} \nu_n(\mathbb{H}) &= 1 \\ \nu(\mathbb{H}) &= 0 \\ \nu_n(\partial\mathbb{H}) &= 0 \\ \nu(\partial\mathbb{H}) &= 1 \end{aligned}$$

But:
If A open then $\nu(A) \leq \liminf_n \nu_n(A)$. If A closed then $\nu(A) \geq \limsup_n \nu_n(A)$.

$$\begin{aligned} C \cdot \nu(A_m) \geq \liminf_n \nu_n(A_m) \geq \nu_\infty(A_m) \geq \nu_\infty(\{x\}) \\ \nu(A_m) \to 0 \text{ as } m \to \infty \\ \nu_n(A_m) = g_n \nu(A_m) = \nu(g_n^{-1} A_m) \leq C \cdot \nu(A_m) \end{aligned}$$

Recall

$$\begin{aligned} \nu_n &= (g_n)_* \nu \\ \nu_\infty &= \lim_{n\to\infty} \nu_n \\ A_m &= \left(-\frac{1}{m} + x, x + \frac{1}{m}\right) \end{aligned}$$

As $m \uparrow \infty$, $\nu(A_m) \to 0$, so $\nu_\infty(\{x\}) = 0$.
Hence,

$$\lim_{n\to\infty} g_n x \text{ exists } \in \partial\mathbb{H}$$

.

9.1 Application

Pick a triangle. Subdivide it into 6 triangles by taking barycentre. Randomly choose 1. Call T_n the n^{th} triangle you get by iterating this procedure.
Claim

$$\frac{\text{area}(T_n)}{\text{diam}(T_n)^2} \to 0 \text{ almost surely}$$

"Triangles get thin ." Space of triangles (up to rotating & rescaling)

$$
\begin{aligned}
X &= \{\tau \in \mathbb{C} : \mathrm{Im}(z) > 0\} \\
B_1 : (0,1,2) &\mapsto \left(0, \frac{1}{2}, \frac{1}{3} + \frac{2}{3}\right) \approx \left(0, 1, \frac{2}{3} + \frac{2\tau}{3}\right) \\
B_2 : (0,1,\tau) &\mapsto \left(\frac{1}{2}, 1, \frac{1}{3} + \frac{2}{3}\right) \approx \left(0, \frac{1}{2}, -\frac{1}{6} + \frac{\tau}{3}\right) \approx \left(0, 1, -\frac{1}{3} + \frac{2\tau}{3}\right) \\
(0,1,2) &\approx \left(0, 1, \frac{-1}{\tau - 1}\right) \\
S(\tau) &= -\frac{1}{\tau - 1} \\
B_3 &= B_1 \circ S
\end{aligned}
$$

Each $B_1, B_2, \ldots, B_6$ is a Mobiös transformation in τ .
$T_0 = (0,1,\tau)$.
$T_n = B_{\epsilon_n} \circ B_{\epsilon_{n-1}} \circ \cdots \circ B_{\epsilon_2} \circ B_{\epsilon_1}(\tau)$.
Check: the measure $\mu = \frac{1}{6}\left(\delta_{B_1} + \delta_{B_2} + \cdots + \delta_{B_6}\right)$ is non-elementary .
Then by Furstenberg's theorem:
a.s. $\widetilde{\tau_n} = B_{\epsilon_1} \circ \cdots \circ B_{\epsilon_n}(\tau) \longrightarrow \xi \in \partial\mathbb{H}$.

$$
\begin{aligned}
\widetilde{T}_n &= (0,1,\widetilde{\tau}_n) \\
\mathrm{area}(T_n) &= \frac{\mathrm{Im} z_n}{2} \\
\mathrm{diam}(T_n) &\geq 1 \\
\frac{\mathrm{area}(\widetilde{T}_n)}{\mathrm{diam}(\widetilde{T}_n)^2} &\leq \frac{1}{2}\mathrm{Im}(\widetilde{\tau}_n)
\end{aligned}
$$

9.2 Observation

n^{th} step distribution of
$B_{\epsilon_1} \cdots B_{\epsilon_n}$ is same as n^{th} step distribution of
$B_{\epsilon_n} \cdots B_{\epsilon_1}$
Hence: since $\widetilde{\tau_n} = B_{\epsilon_1} \circ \cdots \circ B_{\epsilon_n}(\tau) \to \xi \in \partial\mathbb{H}$ a.s.
Then a.s. $\tau_n = B_{\epsilon_n} \circ \cdots \circ B_{\epsilon_1}$
Hence: Im $\tau_n \longrightarrow 0$ as $n \to \infty$.

9.3 Harmonic functions

Let G be a countable group and $\mu \in P(G)$. Then $f : G \to \mathbb{R}$ is μ-harmonic if

$$f(g) = \int_G f(gh)d\mu(h)forallh \in G$$

(mean value property)

Example 9.1. $G = \mathbb{Z}$, $\mu = \frac{1}{2}(\delta_{+1} + \delta_{-1})$

Example 9.2. $f(n) = \frac{1}{2}(f(n+1) + f(n-1))$
Then

$$\underbrace{f(n+1) - f(n)}_{b} = f(n) - f(n-1)$$

$a = f(0)$
Then $f(n) = a + b \cdot n$.
$H^\infty(G,\mu) = \{f : G \to \mathbb{R},\ \text{bounded},\ \mu\text{-harmonic}\}$.
Then $H^\infty(\mathbb{Z}, \mu_{\frac{1}{2}}) = \mathbb{R}$ only constants .

Question: Suppose I give you a finite grid and a function

$$\varphi : \{\text{ boundary nodes }\} \to \mathbb{R}$$

Is there a harmonic function $f : \{\text{ inner nodes}\} \to \mathbb{R}$ which extends φ ? How many?
f harmonic $:= f(x) = \frac{1}{4}\sum_{y\sim x} f(y)$.
Suppose $\varphi \equiv 0 \Rightarrow f = 0$ works .
You can write conditions as a linear system: $\mathbb{R}^{\#\text{nodes}}$. Each condition is a linear equation . So all solutions can be written as $A\vec{x} = b$, where $\vec{x} \in \mathbb{R}^{\#\text{nodes}}$ and b is a boundary condition. Since for $\vec{b} = \vec{0}$ there is only one solution, then A is invertible, hence for <u>any</u> $\vec{b}$ there is a unique solution.
There is a bijection
$\underbrace{\{\text{ boundary data }\}}_{=L^\infty(\partial X)} \leftrightarrow \underbrace{\{\text{ harmonic functions inside }\}}_{=H^\infty(X)}$

10 Harmonic Functions

Let Γ be a graph.

Definition 10.1. A function $f : \Gamma \to \mathbb{R}$ is **harmonic** if

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y).$$

Proposition 10.1. If Γ is a graph with boundary $\partial\Gamma$ then for any function $\phi : \partial\Gamma \to \mathbb{R}$ there is a unique harmonic function $\Phi : \Gamma \to \mathbb{R}$ with $\Gamma\Big|_{\partial\Gamma} = \phi$.

Proof. (Sketch) If $\phi = 0$ then $\Phi = 0$ is an harmonic extension. Moreover by the maximum principle if the maximum of Φ is in int Γ then it must be constant. Similarly, if the minimum of Φ is in int Γ then it must be constant. We conclude that if Φ is not constant then the maximum and the minimum must been the boundary, but this implies $\Phi = 0$. From this we conclude that for any ϕ, Φ is unique since for any other extension Ψ, $\Phi - \Psi = 0$ in the boundary and hence $\Phi = \Psi$.

We can conclude the existence by noticing that harmonicity can be described by a linear map on $\mathbb{R}^n$ where $n = |\text{number of nodes}|$. In this case we have the correspondence between $\phi : \partial\Gamma \to \mathbb{R}$ and solutions x to the system $Ax = b$ where A is the matrix which encodes harmonicity and b the vector of the values of the boundary. □

Remark 10.1. If the graph Γ has no boundary and is finite then $H(G, \mu) = \mathbb{R}$ since all harmonic functions are constant by the maximum principle.

Definition 10.2. If G is group with measure μ a function $f : G \to \mathbb{R}$ is **harmonic** if for all g

$$f(g) = \int_G f(gh) \mathrm{d}\mu(h).$$

Remark 10.2. If G is a finite group with measure μ then $H(G, \mu) = \mathbb{R}$.

What can we say about the harmonic functions of infinite graphs and infinite groups?

Example 10.1. To explore this question let's suppose $G = \mathbb{Z}$ and $\mu = (\delta_{-1} + \delta_1)/2$. In this case we can see that

$$H(\mathbb{Z}, \mu) = \{f(n) = an + b | a, b \in \mathbb{R}\}.$$

by solving the recurrence. Notice that the only bounded harmonic functions are the constant functions.

If instead we let $\mu = (\delta_{-1} + q\delta_1)/(q+1)$, so that one direction is favored more than the other, then μ-harmonicity is equivalent to the equation

$$f(n) = \frac{f(n-1) + qf(n+1)}{q+1},$$

and this has other solutions than are not linear ones. For example if $f(n) = q^{-n}$, then

$$\begin{aligned}\frac{f(n-1) + qf(n+1)}{q+1} &= \frac{q^{1-n} + q(q^{-n-1})}{q+1} \\ &= \frac{q^{1-n} + q^{-n}}{q+1} \\ &= q^{-n} \\ &= f(n),\end{aligned}$$

proving harmonicity.

To prove the equivalent theorem of $\mathbb{R}^2$ we need more definitions and a result.

Definition 10.3. The closed convex hull of a set K is the intersection of the closed convex sets that contain K.

Definition 10.4. An extremal point of K is a point z that is not the nontrivial linear convex combinations of other two points of K, that is, there are no other $x, y \in K$ and an $a \in (0,1)$ with $z = ax + (1-a)y$.

Example 10.2. An important example of the previous definition appears when we consider $T : (X, \mathfrak{A}) \to (X, \mathfrak{A})$ and we define

$$K = \{\mu \in P(X) \mid T_*\mu = \mu\},$$

the set of invariant measures. K is compact in the weak $*$-toplogy and a measure is ergodic if and only if it is an extremal point of K.

Now we can state a theorem that we will use:

Theorem 10.1 (Klein-Milman)**.** Let K be a compact, nonempty, convex subset of a locally convex topological vector space, then K is the closed convex hull of its extremal points.

Now we can say that for $G = \mathbb{Z}^2$ we have the following result:

Theorem 10.2. If $G = \mathbb{R}^2$ and

$$\mu = \frac{\partial_{(1,0)} + \partial_{(-1,0)} + \partial_{(0,1)} + \partial_{(0,-1)}}{4},$$

then there are no nonconstant positive μ-harmonic functions.

Proof. Let $H_1^+(\mathbb{Z}^2) = \{f : \mathbb{Z}^2 \to \mathbb{R} | f \text{ harmonic and positive}, f(0) = 1\} \subset \mathbb{R}^{\mathbb{Z}^2}$. This is a convex compact subset of a locally compact topological vector space (LCTVS). $\mathbb{R}^{\mathbb{Z}^2}$ is a LCTVS but it is not compact.
By induction we can prove that if $f \in H_1^+(\mathbb{Z}^2)$ then

$$f(m,n) \leq 4^{|m|+|n|}.$$

Indeed since

$$f(m,n) = \frac{f(m+1,n) + f(m-1,n) + f(m,n+1) + f(m,n-1)}{4} \geq \frac{f(m+1,n)}{4},$$

then $f(m+1,n) \leq 4f(m,n)$. From this if $y \sim x$ then $f(y) \leq 4f(x)$ and since there is a chain of neighbors that join y with $(0,0)$, whose length is bounded by $|m| + |n|$ we get

$$f(m,n) \leq 4^{|m|+|n|} f(0,0) = 4^{|m|+|n|}.$$

We conclude from here that

$$H_1^+(\mathbb{Z}^2) \subset \prod_{(m,n)\in\mathbb{Z}^2} [0, 4^{|m|+|n|}]$$

which is compact by Tychonoff's theorem. By Klein-Milman it is enough to prove that $H_1^{\text{ext}}(\mathbb{Z}^2) = \{1\}$. We claim that any $h \in H_1^{\text{ext}}(\mathbb{Z}^2)$ is a multiplicative character of $\mathbb{Z}^2$, that is $h(sx) = h(s)h(x)$ for all $s, x \in \mathbb{Z}^2$. Let $s \in S = \{(\pm 1, 0), (0, \pm 1)\}$ and define

$$h_s(x) = \frac{h(sx)}{h(x)}.$$

Notice that $h(x) \neq 0$ always, since otherwise at x we would have a minimum of h and the minimum principle would imply that $h \equiv 0$, since $\mathbb{Z}^2$ has no boundary, and this is not possible since $h(0) = 1$.
We claim that h_s is harmonic. Indeed we have

$$\begin{aligned}
\frac{1}{4}\sum_{t\in S} h_s(xt) &= \frac{1}{4}\sum_{t\in S}\frac{h(xts)}{h(s)} \\
&= \frac{1}{4}\sum_{t\in S}\frac{h(xst)}{h(s)} \\
&= \frac{1}{4h(s)}\sum_{t\in S} h(xst) \\
&= \frac{h(xs)}{h(s)} \\
&= h_s(x)
\end{aligned}$$

proving the claim. Moreover, notice that

$$\frac{1}{4}\sum_{s\in S} h(s)h_s(x) = \frac{1}{4}\sum_{s\in S} h(sx) = h(x),$$

and since

$$\sum_{s\in S}\frac{h(s)}{4} = h(0) = 1,$$

by the harmonicity of h, we conclude that

$$h = \sum_{s\in S}\frac{h(s)}{4}h_s,$$

is a convex linear combination of harmonic functions. Since h is extremal the only way this can happen is that $h = h_s$ for all s, that is,

$$h(sx) = h(s)h(x),$$

as claimed. But there is no multiplicative character, other than 1, on $\mathbb{Z}^2$ since

$$h(s)h(s^{-1}) = h(0) = 1,$$

and so for $s_1 = (1,0)$ and $s_2 = (0,1)$ we have

$$\begin{aligned} 1 &= h(0) \\ &\frac{1}{4}\left(h(s_1) + \frac{1}{h(s_1)} + h(s_2) + \frac{1}{h(s_2)}\right) \\ &\geq 1, \end{aligned}$$

since $\alpha + \alpha^{-1} \geq 2$ with equality iff $\alpha = 1$. We conclude $h(s_1) = h(s_2) = 1$, and since these are the generators, we get $h \equiv 1$. □

Remark 10.3.

1. The same holds for a symmetric random walk on $\mathbb{Z}^d$ with $d \geq 3$.
2. There are multiplicative characters in $\mathbb{Z}$, for example, $n \mapsto q^{-n}$.

Corollary 10.1. There are no bounded nonconstant harmonic functions for $G = \mathbb{Z}^2$.

Proof. Since it is bounded we can translate it by a constant to make it positive and then by previous result it is constant. □

Recall that $f(n) = q^{-n}$ is harmonic for some measure in $\mathbb{Z}$. This motivates the following

Is $H(m,n) = p^{-}mq^{-n}$ μ-harmonic for some measure μ ins $\mathbb{Z}^2$.

We have some results in that direction

Theorem 10.3. If G is non amenable and suppμ generates G as a semigroup then there are non constant bounded harmonic μ-functions on G.

Remark 10.4.

1. $G = \mathbb{Z}^d$ is amenable.
2. There are amenable groups with unbounded nonconstant harmonic functions.

11 The Poisson-Furstenberg boundary

Goal : Let $\mu \in P(G)$.

Find a measure (G,μ) space $(B,0)$. s.t. $H^\infty(G,\mu) \tilde{\to} L^\infty(B,\nu)$

("discrete Poisson rep.formula")

11.1 Ergodic decomposition

Let $T : X \to X$. Define "space of orbits"

Definition 11.1. $x \underset{T}{\sim} y$ if $\exists n,m$ s.t.

$$T^n(x) = T^m(y)$$

We want to construct $Y = X/\underset{T}{\sim}$ so that it is a "nice" measure space .

Definition 11.2. A space X with a σ-algebra $\mathscr{A}$ is *standard* if it is isomorphic (as a Borel space) to ([0,1], Borel sets) .

Definition 11.3. $f : (X,\mathscr{A}) \to (Y,\mathscr{B})$ is Borel isomorphic if it is bijective, f is measurable, and f^{-1} is measurable.

Note: If X is a complete separable metric space then $(X,\text{Borel}) \simeq ([0,1],\text{Borel})$

Lemma 11.1. If M is standard, $A \subset M$. Then A is standard iff is Borel.

Lemma 11.2. The product of two standard spaces is standard.

Definition 11.4. A measure μ as a space S is standard if S is the union of measurable set which is standard and subsets of sets of measure zero.

Definition 11.5. Definition: A standard measure space $(X, \mathscr{A}, \mu)$ is called *Lebesgue* if $\mu(X) = 1$. It is called *smooth Lebesgue* if μ has no atoms.

11.2 Measurable Partitions

Let $(X, \mathscr{A})$ Borel space. A partition of X is a family $\xi = \{\mathcal{C}\}$ of measurable subsets of X which are non-empty, disjoint, and $X = \bigcup_{\mathcal{C} \in \xi} \mathcal{C}$.

If ξ is partition, one can define X/ξ the set of equivalence classes and there is

$$p : X \longrightarrow X/\xi$$

Then you define Borel structure on X/ξ by taking $B \subset X/\xi$ measurable iff $p^{-1}(B)$ is measurable $(p^{-1}(B) \in \mathscr{A})$

Then you define, if $\mu \in P(X)$

$$\nu = p * \mu \in P\left(X/\xi\right)$$

Warning: In general, X/ξ is *not* a standard space!

A family $\mathcal{C}$ of Borel subsets of X separates M if for any $x, y \in X$ with $x \neq y, \exists B \in \mathcal{C}$ such that either $x \in B, y \notin B$ or $x \notin B, y \in B$.

X is countably separated if there exists a countable family of Borel subsets which separates points in X .

Example 11.1. $([0,1], \text{Borel})$. The family of balls of rational center and rational radius separates points.

Example 11.2. $X = S^1$ $T_\alpha(x) = \overset{x+\alpha}{(\text{mod} 1)}, \alpha \notin \mathbb{Q}$. Take $Y = X/\xi$ where ξ are T_α-orbits. Then Y is not countable separated by $\mathscr{B} = \mathscr{A}/\xi$. $L^\infty\left(X/\xi, \mathscr{B}/\xi\right)$ is not separable.

Definition 11.6. A partition ξ is measurable if X/ξ is countably separated.

Theorem 11.1. If $(X, \mathscr{A}, \mu)$ a Lebesgue space and ξ is a measurable partition, then $(Y, \mathscr{B}, \nu)$ where

$$Y = {}^{X}\!/_{\xi}$$
$$\mathscr{B} = {}^{\mathscr{A}}\!/_{\xi}$$
$$\nu = p * \mu$$

is Lebesgue.

Given a partition ξ on X, its measurable envelope is the finest measurable partition $\hat{\xi}$ s.t. ξ is a refinement of $\hat{\xi}$. This always exists in a Lebesgue space, and it is unique up to sets of measure 0. (cf. Rohlin)

Example 11.3. $T_\alpha : S^1 \to S^1$. $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. The measurable envelope of ξ_T is the trivial partition (X, ϕ) .

11.3 Space of Ergodic Components

Let $(X, \mathscr{A}, \mu)$ a Lebesgue space, T mps. Then the quotient of X by the measurable envelope of $\sim T$ is called
${}^{X}/\!/_{T}$ the space of ergodic components
${}^{X}/\!/_{T}$ is endowed with σ-algebra $\mathscr{B}$ and measure $\gamma = p_*(\mu)$

11.4 Properties

There is a map $p : X \to {}^{X}/\!/_{T}$ s.t. $p \circ T = p$.
For ever measurable $f : X \to Z$ s.t. $f \circ T = f$ there exists $g : {}^{X}/\!/_{T} \to Z$ s.t. $f = g \circ p$.
In particular: $L^\infty(X, \mathscr{A}, \mu)^T \simeq L^\infty({}^{X}/\!/_{T}, \mathscr{B}, \nu)$

11.5 Ergodic Decomposition

Let $T : (X, \mathscr{A}, \mu) \to (X, \mathscr{A}, \mu) \quad (T_* \mu = \mu, X \text{ standard})$. Then there is a map $p : (X, \mu) \to (Y, \mu)$ and measure μ_y for each y s.t.

$$\mu = \int_Y \mu_y \, d\nu(y)$$

and so that μ_y is T-invariant and ergodic for almost all $y \in Y$.
Note that $Y = {}^{X}/\!/_{T}$.
If μ is ergodic for T, then $Y = \{\text{pt}\}$, $\mu = \mu_y$.

If T is ergodic for μ

$$L^\infty(X, \mathscr{A}, \mu)^T = \mathbb{R} = L^\infty(\{\text{pt}\})$$
$$\Rightarrow X /\!/ T = \{\text{pt}\}$$

Definition 11.7 (Construction of Poisson-Furstenberg boundary)**.**
$\Omega = G^{\mathbb{N}}$ space of sample paths .
$T : \Omega \to \Omega$ time shift .
$G \to \Omega$
$g \cdot (\xi_0, \xi_1, \dots,) = (g\xi_0, g\xi_1, \dots)$
$\mathbb{P}$ on Ω distribution of RW starting at 1.
$\delta_e \times \mu^{\mathbb{N}}$
$G \times G^{\mathbb{N}} \to G^{\mathbb{N}} = \Omega$ T
$(\omega_0, \omega_1, \dots) \mapsto (\omega_0, \omega_0\omega_1, \omega_0\omega_1\omega_2, \dots)$
$\omega_0 = \xi_0$
$\omega_0\omega_1 = \xi_1$
$\omega_0\omega_1\omega_2 = \xi_2$
(step space)
The Poisson-Furstenberg boundary of (G, μ) is the space of ergodic components of $(\Omega, \mathbb{P})$ w.r.t. $T : \Omega \to \Omega$.

Note: One should check that $(\Omega, \mathbb{P})$ is Lebesgue. This is certainly true if G is countable. (Also if G is locally compact 2^{nd} countable.)
There is a boundary map $\text{bnd} : (\Omega, \mathbb{P}) \longrightarrow (B_{\text{PF}}, \nu_{\text{PF}})$.
Where $B_{\text{PF}} = (\Omega, \mathbb{P}) /\!/ T$.
$\nu_{\text{PF}} = \text{bnd}_*(\mathbb{P})$.
$\text{bnd} \circ T = \text{bnd}$.
bnd is G-equivalent .

Lemma 11.3. ν_{PF} is μ-stationary .

Proof. $\mu * \nu_{\text{PF}} \stackrel{?}{=} \nu_{\text{PF}}$
$\nu_{\text{PF}} = \text{bnd}_*(\mathbb{P}) = \text{bnd}_* T_* \mathbb{P} = \mu * \text{bnd}_*(\mathbb{P})$
$T_*\mathbb{P} = \mathbb{P}_\mu$ (distribution of RW qith steps $\mu^{\mathbb{N}}$,
$\text{bnd}_*(\mathbb{P}_\mu) \stackrel{?}{=} \mu * \text{bnd}_*(\mathbb{P})$
$\mu = \sum \mu(g)\delta_g$
$\varphi_*(\mu \times \mu^{\mathbb{N}}) = \varphi_*(\sum \mu(g)\delta_g \times \mu^{\mathbb{N}}) = \sum \mu(g)\varphi_*(\delta_g \times \mu^{\mathbb{N}})$
$\mu * \text{bnd}_*(\mathbb{P}) = \sum \mu(g) g_* \text{bnd}_*(\mathbb{P}) = \sum \mu(g) \text{bnd}_*(g_*\mathbb{P})$

$\mathbb{P} = \varphi_*(\delta_e \times \mu^{\mathbb{N}})$

$$\begin{aligned} g_*\mathbb{P} &= g_*\varphi_*(\delta_e \times \mu^{\mathbb{N}}) \\ &= \varphi_* g_*(\delta_e \times \mu^{\mathbb{N}}) \\ &= \varphi_*(\delta_g \times \mu^{\mathbb{N}}) \end{aligned}$$

□

$Note : G : G \times G^{\mathbb{N}}$
$g \cdot (\omega_0, \omega_1, \ldots) = (g\omega_0, \omega_1, \ldots)$

12 The Poisson-Furstenberg boundary II

G, μ
$\Omega = G^{\mathbb{N}}$ sample paths
$\mathbb{P}$
$T : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ time shift
$x \underset{T}{\sim} y \iff \exists n, m$ such that $T^n x = T^m y$

Definition 12.1. $(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) := {}^{(\Omega, \mathbb{P})}/\!/_{T}$

$(L^\infty(\Omega, (\mathbb{P}))^T \simeq L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$
$\mathrm{bnd} : (\Omega, \mathbb{P}) \longrightarrow (B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$

1. $\mathrm{bnd} \circ T = \mathrm{bnd}$

2. If $f : (\Omega, \mathbb{P}) \to (Y, \lambda)$ measurable, $f \circ T = f$, then there exists $g : (B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \to (Y, \lambda)$ such that $f = g \circ \mathrm{bnd}$

For which (G, μ) is B_{PF} "trivial"?
i.e. $L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \simeq L^\infty(\{\mathrm{pt}\}) = \mathbb{R}$.
Given an action of G on (M, λ) , is M a "model" for PF boundary, i.e. $L^\infty(M, \lambda) \simeq L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$?
(Where the isomorphisms are G-equivariant.)
Note: B_{PF} is a G-space, bnd is G-equivariant, and ν_{PF} is μ-stationary.

Theorem 12.1. There is an isometric isomorphism

$$L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \xrightarrow{\sim} H^\infty(G, \mu)$$

.

This is an analogue of the Poisson representation formula.

Remark 12.1. $(B_{\text{PF}}, \nu_{\text{PF}})$ is trivial $\iff$ the only bdd harmonic functions are constant .

12.1 Poisson transform

Let (B, ν) be a (G, μ)-space.
Define, for $f \in L^\infty(B, \nu)$ the Poisson transform

$$\mathcal{P}f(g) := \int_B f(gx)\ d\nu(x) \text{ for } g \in G.$$

Lemma 12.1. $\mathcal{P}f$ is μ-harmonic.

Proof.

$$\begin{aligned}
\mathcal{P}f(g) &\overset{?}{=} \int_G \mathcal{P}f(gh)\ d\mu(h) \\
&= \int_G \int_B f(ghx)\ d\nu(x) d\mu(h) \\
&= \int_B f(gy)\ d(\mu * \nu)(y) \\
&= \int_B f(gy)\ d\nu(y) \\
&= \mathcal{P}f(g)
\end{aligned}$$

(where $y = hx$)
This gives the map:
$\mathcal{P}f : L^\infty(B_{\text{PF}}, \nu_{\text{PF}}) \to H^\infty(G, \mu)$

□

Note: If (B, ν) is a (G, μ)-space, there is a G-equivariant map φ

$$\varphi : (B_{\text{PF}}, \nu_{\text{PF}}) \to (B, \nu)$$

This is equivalent to constructing a T-invariant $f : (\Omega, \mathbb{P}) \to (B, \nu)$.
Pick $(g_n) \in \Omega$, pick $\varphi : B \to \mathbb{R}$ measurable.
$X_n = \int_B \varphi(g_n x)\ d\nu(x)$
Claim: X_n is a martingale.
$(g_n = \omega_0 \omega_1 \cdots \omega_n)$

$\mathbb{E}[x_{n+1}|X_n] = \int_B \varphi(\omega_0 \cdots \omega_n \underbrace{\omega_{n+1} x}_{y})\ d\nu(x) d\mu(\omega_{n+1}) = \int_B \varphi(\omega_0 \cdots \omega_n y)\ d\nu(y) =$
X_n .
By the *martingale convergence theorem* : $\lim_n X_n = X_\infty$ exists $\mathbb{P}$ a.s.
By construction, X_∞ is T-invariant.
$L^\infty(B,\nu) \longrightarrow L^\infty(\Omega,\mathbb{P})^T \simeq L^\infty(B_{\mathrm{PF}},\nu_{\mathrm{PF}})$
$\varphi \longmapsto X_\infty(\varphi)$
We produced a G-equivariant map.
$L^\infty(B,\nu) \longrightarrow L^\infty(B_{\mathrm{PF}},\nu_{\mathrm{PF}})$.
By duality, this gives a G-equivariant :
$(B_{\mathrm{PF}},\nu_{\mathrm{PF}}) \longrightarrow (B,\nu)$
Why is X_∞ G-equivariant?
$X_n \circ T = \int_B \varphi(g_{n+1}x)\ d\nu(x)$
$\lim_{n\to\infty} \int_B \varphi(g_{n+1}x)\ d\nu(x) = \lim_{n\to\infty} \int_B \varphi(g_n x)\ d\nu(x)$

12.2 Geometric intuition

Poisson transform $\longleftrightarrow$ convolution with Poisson kernel .
$f : S^1 \to \mathbb{R}$ bdd, measurable.
$\varphi(z) = \int_{S^1} \underbrace{K(z,\xi)}_{\text{Poisson kernel}}\ dm(\xi)$
in $\mathbb{H}$ model.
Inverse of Poisson transform $\longleftrightarrow$ $K(z,\infty) = \mathrm{Im}(z)$ taking limit as $z \to \partial\mathbb{D}$ same if you take limit along Brownian path.
In general:

- $K(z,\xi)$ is harmonic in z
- $\lim_{z\to\xi} K(z,\xi) = +\infty$
- $\lim_{z\to y\neq\xi} k(z,\xi) = 0$

Let $f \in \mathbb{R}^\infty(\mathbb{D})$, take $x \in \mathbb{D}$. Let ω_t be a Brownian path in $\mathbb{D}$ s.t. $\omega_0 = x$.
Then take

$$\lim_{t\to\infty} f(\omega_t) = \widehat{f}(\xi)$$

where $\xi = \lim_{t\to\infty} \omega_t$.
Strategy of proof of theorem

$\varphi \in H^\infty(G,\mu)$
Define $f \in L^\infty(B,\nu)$.
Pick (g_n) s.t. $g_n x \to \xi$.
Consider $\lim\limits_{n\to\infty} \varphi(g_n x) = f(\xi)$.

12.3 Proof of the Theorem

We have a map $P : L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \to H^\infty(G,\mu)$. Let us define a map

$$\Lambda : H^\infty(G,\mu) \to L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$$

Let $h \in H^\infty(G,\mu)$, pick $(g_n) \in \Omega$.
$\widehat{h}(g_0, g_1, \ldots) \coloneqq \lim\limits_{n\to\infty} h(g_n)$ (is T-invariant exists by MCT)
Hence it defines $\widehat{h} \in L^\infty(\Omega,\mathbb{P})^T = L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$.
Note:
$||P|| \leq 1$
$||\Lambda|| \leq 1$
Hence if we prove $P \circ \Lambda = \Lambda \circ P = \mathrm{Id}$ then P and Λ are isometries.
Check:
$f \in L^\infty(B_{\mathrm{PF}}) = L^\infty(\Omega)^T$
$\widehat{Pf} \stackrel{?}{=} f$

$$\begin{aligned}\widehat{Pf}(g_n) &= \lim_{n\to\infty} Pf(g_n) = \lim_{n\to\infty} \int_\Omega f(g_n \cdot \omega')\ d\mathbb{P}(\omega') = \lim_{n\to\infty} \int_\Omega f(\overbrace{g_n\omega'_0, \omega'_1, \ldots}^{\text{increments}})\ d\mathbb{P}(\omega') \\ &= \lim_{n\to\infty} \int_\Omega f \circ \cancel{T^n}(\omega_0, \omega_1, \ldots, \omega_n)\ d\mathbb{P}(\omega') \stackrel{?}{=} f(\omega_0, \ldots, \omega_n, \ldots)\end{aligned}$$

If f depends only on a finite number of variables then equality.
Otherwise, you have to approximate f with such functions, which are dense.

13 The Poisson-Furstenberg boundary III

Theorem 13.1. There is an isometric isomorphism:

$$L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \xrightarrow{\sim} H^\infty(G, \mu)$$

Proof. $L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) = (L(\Omega, \mathbb{P}))^T$.
Poisson transform: $\mathcal{P} : L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \longrightarrow H^\infty(G, \mu)$
"convolution w. Poisson kernel"
$\mathcal{P}f(g) = \int_{B_{\mathrm{PF}}} f(gx) d\nu(x)$
Poisson kernel: $K_g(x) = \frac{dg\nu}{d\nu}(x)$
$g \in SL_2\mathbb{R} \hookrightarrow \partial\mathbb{H}^2 \qquad \nu =$ Lebesgue
Inverse Poisson Transform
$\Lambda : H^\infty(G, \mu) \to L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}})$
"taking limit of Brownian motion"
$[(g_n)] \in B_{\mathrm{PF}}$ with $(g_n) \in \Omega$
$\Lambda h((g_n)) \coloneqq \lim_{n\to\infty} h(g_n)$ (a.s. exists)
$\mathcal{P} \circ \Lambda \stackrel{?}{=} \mathrm{id}$
$\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ (increments)
$g_n = \omega_0 \cdots \omega_n$
$g\omega = (g\omega_0, \omega_1, \ldots, \omega_n, \ldots)$

$$\begin{aligned}
\mathcal{P}(\Lambda(h))(g) &= \int_\Omega \Lambda h(g\omega)\ d\mu^{\mathbb{N}}(\omega) \\
&= \int_\Omega \lim_{n\to\infty} h(gg_n)\ d\mu^{\mathbb{N}}(\omega) \\
&= \lim_{n\to\infty} \int_\Omega h(gg_n)\ d\mu^{\mathbb{N}}(\omega) \qquad (h \text{ is bounded }) \\
&= \lim_{n\to\infty} \int_\Omega h(g\omega_0\cdots\omega_{n-1}\omega_n)\ d\mu^{\mathbb{N}}(\omega_0, \ldots, \omega_{n-1}) \otimes\ d\mu(\omega_n) \\
&= \lim_{n\to\infty} \int_\Omega h(g\omega_0\cdots\omega_{n-1})\ d\mu^{\mathbb{N}}(\omega_0, \ldots, \omega_{n-1}) \\
&= \lim_{n\to\infty} \int_\Omega h(g) = h(g)\ \checkmark
\end{aligned}$$

□

13.1 Poisson-Furstenberg entropy

Question: Can we find a geometric model for B_{PF}?

Is B_{PF} trivial?

- entropy of RW
- entropy of boundary (Furstenberg entropy)

Let μ be a measure on a finite (or countable) set X .

Definition 13.1. $H(\mu) = -\sum_{x \in X} \mu(x) \log \mu(x)$

Idea: $\varphi(x) = -\log \mu(x)$ is the "quantity of information at x"

Entropy is "average quantity of information"

$H(\mu) = \mathbb{E}[\varphi(x)]$

Let X be finite, $|X| = N$.

For what μ is $H(\mu)$ max?

Lemma 13.1. $H(\mu) \leq H(\mu_{\text{unif}})$ (where μ_{unif} is the uniform distribution on X .)

Proof. $H(\mu) = \mathbb{E}_\mu(-\log \mu)$

Jensen: If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex then $\varphi(\mathbb{E}[f]) \leq \mathbb{E}(\varphi \circ f)$

$$\sum_i^k \mu_i(-\log \mu_i)$$

$\varphi(x_1, \ldots, x_n) = \sum_i^k -\log x_i$ (sum of convex functions is convex)

$$\mathbb{E}_\mu[-\log \mu] = \mathbb{E}_\mu\left[\log\left[\frac{1}{\mu}\right]\right] \leq \log \mathbb{E}_\mu\left[\frac{1}{\mu}\right] = \log N$$

$$\mathbb{E}_\mu\left(\frac{1}{\mu}\right) = N \ .$$

$\mu_{\text{inf}} = \frac{1}{N}$

□

14 Variational Principle

The uniform measure gives you maximal entropy. The topological entropy is the maximum of all measure-theoretic entropies.

Definition 14.1 (Entropy of RW)**.** Let $\mu \in P(G)$. Then the (asymptotic) entropy of RW with step distribution μ is

$$H_\infty(\mu) := \lim_{n\to\infty} \frac{H(\mu^(m))}{n} \quad \circledast$$

where $\mu^{(n)} = \underbrace{\mu \star \mu \star \cdots \star \mu}_{n}$.

(n^{th} step distribution of RW)

Note: $0 \le H(\mu) \le \log|X|$ (if $|X|$ is finite)

Properties

1. $H(\mu_1 \otimes \mu_2) = H(\mu_1) + H(\mu_2)$
2. If $\Pi : X \longrightarrow Y$ is surjective and $\nu = \Pi_* \mu$ then $H(\nu) \le H(\mu)$ (exercise)

$H(\mu^{(n+1)}) = H(\Pi_*(\mu^{(n)} \otimes \mu)) \le H(\mu^{(n)} \otimes \mu)$

$\Pi : G \times G \longrightarrow G$

$\mu^{(n)}, \mu, \mu^{(n+1)}$

The sequence $(H(\mu^{(n)})_{n\in\mathbb{N}}$ is sub-additive, hence a limit $\circledast$ exists (could be infinite if $|\text{supp}\mu| < \infty$.

If $H(\mu) < \infty$, then $H_\infty(\mu) < \infty$.

(Such measures are called of *finite entropy* .)

15 Entropy of a Boundary

Let (B,ν) be a (G,μ)-space.

Definition 15.1.

$$\rho : G \times B \to \mathbb{R} the Radon-Nikodyn cocycle. \rho(g,x) \quad := -\log\left[\frac{dg^{-1}\nu}{d\nu}(x)\right]$$

Lemma 15.1. RN cocycle is an additive cocycle.

$\rho(gg', x) = \rho(g, g'x) + \rho(g', x) \quad \forall g, g' \in G \ \forall x \in X$
The entropy of (B, ν) is
$h_\mu(B, \nu) = \int_{G\times B} \rho(g, x) \ d\mu(g) d\nu(x)$.

Lemma 15.2. $h_\mu(B, \nu) \geq 0$ and
$h_\mu = 0 \iff \nu$ is G-invariant (if $g =< \text{supp}, \mu >$)

Proof.

$$
\begin{aligned}
h_\mu(B, \nu) &= \int_G \ d\mu(g) \int_B -\log\left(\frac{dg^{-1}\nu}{d\nu}(x)\right) \ d\nu(x) \\
&\overset{\text{Jensen}}{\geq} \int_G \ d\mu(g) \left(-\log\left(\int_B \frac{dg^{-1}\nu}{d\nu}(x) \ d\nu(x)\right)\right) \\
&= \int_G \ d\mu(g)(-\log(1)) = 0 \checkmark
\end{aligned}
$$

$-\log$ is strictly convex, so:
Equality $\implies \dfrac{dg^{-1}\nu}{d\nu}(x) = 1 \implies g^{-1}\nu = \nu \ \forall g \in \text{supp}\mu \implies \nu$ is G-invariant
(for μ- a.e. g) ☐

G has subexponential growth if there is a finite generating set S such that

$$
\lim_{R\to\infty} \frac{\log \#\{g \in G : \|g\|_S \leq R\}}{R} = 0
$$

$\|g\|_S = \min\{k : g = s_1^{\pm} \cdots s_k^{\pm}, \text{ where } s_i \in S\}$
WORD METRIC
The word metric changes with S but the fact of having subexponential growth does *not*.

Proposition 15.1. If G has subexponential growth and μ is finitely supported, then $H_\infty(\mu) = 0$.

Proof. $H(\mu^{(n)}) \leq \log \#\text{supp}(\mu^{(n)}) \leq \log \#B(1, Cn)$

By dividing by n you get $\lim_{n\to\infty} \dfrac{H(\mu^{(n)})}{n} = 0$.
(C depends on supp μ)

☐

16 Boundaries

Let (B,ν) a (G,μ)-space. (Assume $G = \langle \text{supp}\mu\rangle_{\text{sgr}}$) .

$$h_\mu(B,\nu) = \int_B \int_G -\log \frac{dg_\star^{-1}\nu}{d\nu}(x)\ d\mu(g)\ d\nu(x)$$
$$0 \le h_\mu(B,\nu)$$
$$h_\mu(B,\nu) = 0 \iff \nu \text{ is } G\text{-invariant.}$$
$$H_\infty(\mu) = \lim_{n\to\infty} \frac{H(\mu^{(n)}}{n} (\text{entropy of RW})$$

Lemma 16.1. For every (B,ν), $0 \le h_\mu(B,\nu) \le H_\infty(\mu)$

Proof. $\alpha,\beta \in P(G)$, $\nu \in P(B)$.
$h_{\alpha\star\beta}(B,\nu) = h_\alpha(B,\beta\star\nu) + h_\beta(B,\nu)$
Recall: $\rho(g,x) = -\log \frac{dg^{-1}\nu}{d\nu}(x)$ is a cocycle.
$\rho(gg',x) = \rho(g,g'x) + \rho(g',x)$.
Integrate
$\int \rho(gg',x)\ d\alpha(g)d\beta(g')d\nu(x) = \int \rho(g,g'x)\ d\alpha(g)d\beta(g')d\nu(x) + \int \rho(g',x)\ d\alpha(g)d\beta(g')d\nu(x)$
$h = gg'$ and $y = g'x$
$\int \rho(h,x)\ d(\alpha\star\beta)(h)d\nu(x) = h_{\alpha\star\beta}(B,\nu)$
$\int \rho(g,y)\ d\alpha(g)d(\beta\star\nu)(y) + h_\beta(B,\nu) = h_\alpha(B,\beta\star\nu) + h_\beta(B,\nu)$
If $\alpha = \beta = \mu$ then
$h_{\mu\star\mu}(B,\nu) = h_\mu(B,\mu\star\nu) + h_\mu(B,\nu)$
$h_{\mu^{(n)}}(B,\nu) = 2\cdot h_\mu(B,\nu)$
For $n \ge 1$
$h_{\mu^{(n}}(B,\nu) = \eta\cdot h_\mu(B,\nu)$
$\nu = \sum \mu(g')g'_\star\nu \ge \mu(g)g_\star\nu \quad \forall g \in G$.
for $g \in \text{supp}\mu$
$g_\star\nu \le \frac{1}{\mu(g)}\nu \implies$ for ν a.e. $x \qquad \frac{dg_\star\nu}{d\nu}(x) \le \frac{1}{\mu(g)}$
$\rho(g,x) = -\rho(g^{-1},gx) = \log \frac{dg_\star\nu}{d\nu}(gx) \le \log\left(\frac{1}{\mu(g)}\right) = -\log\mu(g)$
$\rho(g^{-1},gx) + \rho(g,x) = \rho(1,x) = 0$

By integrating:
$h_\mu(B,\nu) \le H(\mu)$
Applying to $\mu^{(n)}$
$\underbrace{h_\mu(n)(B,\nu)}_{=nh_\mu(B,\nu)} \le H(\mu^{(n)})$

$h_\mu(B,\nu) \le \dfrac{H(\mu^{(n)})}{n} \to H_\infty(\mu)$ ✓

□

Corollary 16.1. $H_\infty(\mu) = 0 \implies$ every μ-stationary measure is G-invariant

Theorem 16.1. (entropy criterion, Kaimanovich-Vershik, Devionic, Rosenblatt) Let μ be a prob. on countable G, $G = \langle \mathrm{supp}\mu \rangle_{\mathrm{sgr}}$, and $H_\infty(\mu) < \infty$. Then the Poisson boundary of (G,μ) is trivial iff $H_\infty(\mu) = 0$.
(by definition this is equivalent to the fact that every bounded μ-harmonic function is constant).

Proof. (1)
$0 \le h_\mu(B,\nu) \le H_\infty(\mu)$
If $H_\infty(\mu) = 0 \implies h_\mu(B\nu)$ for every $B \implies h_\mu(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) = 0$
Note: If (B,ν) is (G,μ)-space, then thre is a G-equivalent
$\varphi : (B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \to (B,\nu)$.
So $h_\mu(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \ge h_\mu(B,\nu)$
ν_{PF} is G-invariant.
$L^\infty(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \xrightarrow{\mathcal{P}} H^\infty(G,\mu)$ if ν_{PF} is G-invariant.
$\mathcal{P}f(g) = \int f \; dg\nu = \int f \; d\nu$
$\mathcal{P}f$ is <u>constant</u> .
Then: every bdd harmonic function is in the image of $\mathcal{P}$, hence constant.
$H_\infty(\mu) = 0 \longleftrightarrow h_\mu(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) = 0 \longleftrightarrow \nu_{\mathrm{PF}}$ is G-invariant $\longleftrightarrow$ every bdd harmonic is constant

□

Proof. (2)
Let $(\Omega, \mathbb{P})$ space of sample paths.
$g_n = \omega_0\omega_1\cdots\omega_n$
$(g) \overset{\eta_n}{\sim} (g') := g_k = g'_k \qquad \forall k \ge n$.
$(g) \overset{\alpha_n}{\sim} (g') := g_k = g'_k \qquad \forall k \le n$.
Equivalence classes for α_n (any n) are called <u>cylinders</u> .

$\alpha_{n+1} > \alpha_n$ (finer)
$\eta_{n+1} < \eta_n$
$\bigvee_{n=1}^{\infty} \alpha_n = P$ (all subsets)
$\bigwedge_{n=1}^{\infty} \eta_n = \mathcal{S}$ (tail σ-algebra)
E.g. "there are 100 heads infinitely often"
By definition: $(B_{\mathrm{PF}}, \nu_{\mathrm{PF}}) \simeq (\Omega, \mathcal{S}, \mathbb{P}|_{\mathcal{S}})$.
Poisson ∂ trivial $\longleftrightarrow$ Tail events have prob $= 0, 1$. □

16.1 Conditional Entropy

Let ξ, η be partitions. The *conditional entropy* (in a Lebesgue space (X, m)) of ξ w.r.t. η is
$H(\xi|\eta) = -\int \log m(x, \xi|\eta)\ dm(x)$.
$m(x, \xi|\eta)$ is conditional measure of $\xi(x)$ w.r.t. η .
$\xi(x)$ = element of ξ which contains x
$m(x, \xi|\eta) = \dfrac{m(\xi(x) \cap \eta(x))}{m(\eta(x))}$
Suppose ξ, η are independent.
$m(\xi(x) \cap \eta(x)) = m(\xi(x)) \cdot m(\eta(x))$
$\implies m(x, \xi|\eta) = m(\xi(x))$
$H(\xi|\eta) = H(\xi) \iff \xi, \eta$ are independent.
If $\eta_{n+1} < \eta_n \qquad \forall n \qquad \eta_\infty = \bigwedge_{n=1}^{\infty} \eta_n$
then $H(\xi|\eta_\infty) = \lim_{n\to\infty} H(\xi|\eta_n)$.

Proposition 16.1. For $0 \le k \le n$, $H(\alpha_k|\eta_n) = kh_1 + h_{n-k} - h_n$.

Proof.
$$\mathbb{P}(g, \alpha_k|\eta_n) = \frac{\mathbb{P}(\mathcal{C}_g^{\overline{0,k}} \cap \mathcal{C}_g^n)}{\mathbb{P}(\mathcal{C}_g^n)} = \frac{\mu(\omega_0)\cdots\mu(\omega_k) \cdot \mu_{n-k}(g_k^{-1} g_n)}{\mu_n(g_n)} .$$
Note: $g_n = \underbrace{\omega_0\cdots\omega_k}_{g_k} \cdot \underbrace{\omega_{k+1}\cdots\omega_n}_{g_k^{-1} g_n}$
$\implies kH(\mu) + H(\mu^{(n-k)}) - H(\mu^{(n)}) = H(\alpha_k|\eta_n)$
$H(\alpha_1|\eta_n) = h_1 + h_{n-1} - h_n$
Since $\eta_n > \eta_{n+1}$
$H(\alpha_1|\eta_n) \le H(\alpha_1|\eta_{n+1})$

$$H(\alpha_1|\eta) = H(\mu) - \underbrace{h(G,\mu)}_{\text{entropy of RW}}$$

□

17 Identification of the Poisson Boundary

$(G,\mu) \hookrightarrow (X,d)$

There is a geometric boundary ∂X .

Is ∂X a model for the PF boundary of (G,μ)?

Step 1

Prove that RW on $G \hookrightarrow X$ converges to ∂X .

If so, then define *hitting measure* ν on ∂X .

$\nu(Z) = \mathbb{P}\left(\lim_{n\to\infty} w_n x \in A\right)$

Example 17.1. $G < SL_2(\mathbb{R}) \to X = \mathbb{H}^2$

$\partial X = S^1$

$G = G_k$

$\partial X =$

Then $(\partial X,\nu)$ is a (G,μ) - space because hitting measure is stationary.

This implies that that there is a map

$\varphi : \overset{\circlearrowright T}{(\Omega,\mathbb{P})} \to (\partial X,\nu)$

$\varphi((g_n)) := \lim_{n\to\infty} g_n x \in \partial X$.

By construction: $\varphi \circ \underset{=\text{shift}}{T} = \varphi$

Hence: there is a map $\psi : (B_{\text{PF}}, \nu_{\text{PF})} \to (\partial X, 0)$ such that

$(\partial X,\nu)$ is a "quotient" of $(B_{\text{PF}},\nu_{\text{PF}})$.

(B,ν) is a (G,μ)-boundary if there exists a G-equivalent $p : (B_{\text{PF}}, \nu_{\text{PF})} \to (B,\nu)$.

Step 2

Prove that $(\partial X,\nu)$ is maximal.

Let μ be of finite entropy.

Theorem 17.1. A (G,μ)-boundary (B,ν) is isomorphic to $(B_{\text{PF}},\nu_{\text{PF}})$ if and only if $h(\mathbb{P}^\xi) = 0$ for ν-a.e. $\xi \in B$.

Where $\mathbb{P}^\xi$ is the conditional measure of $\mathbb{P}$ w.r.t. event $A_\xi = \{\text{bnd}(g_n) = \xi\}$.

Recall bnd$(\Omega, \mathbb{P}) \to (B, \nu)$ is given by definition of (G, μ)-boundary .
Conditions to identify Poisson boundary
- ray approximation
- strip approximation
Ray approximation $\longleftrightarrow$ Sub-linear tracking
Law of Large Numbers
$(X_n) : \Omega \to \mathbb{R}$ iid $\mathbb{E}[X_1] = \ell < \infty$.
Then a.s. $\dfrac{X_1(\omega) + \cdots + X_n(\omega)}{n} = \ell$ $\circledast$
Generalize to non-abelian groups?
$\circledast \longleftrightarrow \dfrac{|X_1(\omega) + \cdots + X_n(\omega) - \ell_n|}{n} \to 0$
for some $\gamma : [0, \infty) \to \mathbb{R}$ geodesic ray .
Let $(G, \mu) \to (X, d)$.
The measure μ has finite 1^{st} moment if

$$\int_G d(gx, x)\ d\mu(g) < \infty$$

We say the RW has the sub-linear tracking property if $\exists \ell \geq 0, \exists x \in X$ s.t. for a.e. sample path (w_n) there exists a geodesic ray

$$\gamma : [0, \infty) \to X \text{ s.t.}$$
$$\frac{d(w_n x, \gamma(\ell_n)}{n} \longrightarrow 0$$

We say that the action of $G \to X$ has exponentially bounded growth if $\exists C \geq 0$ s.t.
$\#\{g \in G : d(gx, x) \leq R\} \leq e^{Ck}$ for all $R > 0$

Theorem 17.2 (Kaimonivich's ray criterion)**.** Let G be a countable group of isometries of (X, d) geodesic metric space, and let $\overline{X} = X \partial X$ be a compactification of X. Let μ on G have finite first moment, and action be of exponential bounded growth. Then if $x \in X$, and

1. a.e. sample path $(w_n x)$ converges to ∂X
2. The sublinear tracking property holds

Then $(\partial X, \nu)$ is isomorphic to the Poisson-Furstenberg boundary of (G, μ).

History:
Kaimanovich (hyperbolic groups)

Karlsson - Margulis (G acts on a CAT(0) space)
A geodesic metric space (X,d) is δ-hyperbolic if for any $x,y,z\in X$,
$[x,y]\subseteq N_\delta([x,z]\cup N_\delta([y,z])$

Exercise 17.1. Prove that $(\mathbb{H}^2,d_{\text{hyp}})$ is δ-hyperbolic. What is δ =?

$SL_2(\mathbb{R})$ acts transitively on the set of triples of $\partial\mathbb{H}^2\simeq\mathbb{P}^1(\mathbb{R})_\infty$
Given $K\leq 0$ a metric space is CAT(K) if for any $x,y,z\in X$ the following holds: consider the space
$\mathbb{H}_k$ constant curvature $\equiv K$.
Construct a comparison triangle $\widetilde{T}_k(x,y,z)$ s.t.
$d_{\mathbb{H}_k}(\widetilde{x},\widetilde{y})=d_X(x,y)$
$d_{\mathbb{H}_k}(\widetilde{x},\widetilde{z})=d_X(x,z)$
$d_{\mathbb{H}_k}(\widetilde{y},\widetilde{z})=d_X(y,z)$
Then for any $v,w\in T$ with corresponding $\widetilde{v},\widetilde{w}\in\widetilde{T}$.
$d_X(v,w)\leq d_{\mathbb{H}^k}(\widetilde{v},\widetilde{w})$
$X=\mathbb{H}^2\cup\bigcup_i^r S_i$ (S_i spheres)
In particular:
CAT(-1) $\longrightarrow$ hyperbolic
CAT(0) $\longrightarrow$ "either hyperbolic or flat"
Difference: CAT(0) is "local" property
Hyperbolic is "global" (large scale geometry)

Definition 17.1. (X,d) is proper if closed balls in X are compact .

$\overline{B}(x_0,R)=\{x:d(x,x_0)\leq R\}$

Example 17.2. Graph with bounded valence G = finitely generated group X = Cayley graph

Example 17.3. X = finitely dimensional manifold .

Example 17.4. X = locally infinite graph (NON-example)

If X is not proper, then there is no compactification $\overline{X}=X\cup\partial X$.
Typically, ∂X will not be compact .
Remark: There are non-proper, δ-hyperbolic spaces .

Example 17.5. A tree with ∞ valence is δ-hyperbolic with $\delta=0$

Example 17.6. The curve complex of a closed surface S is δ-hyperbolic, non-proper.

18 Sublinear tracking

$(G,\mu) \to (X,d)$
X geodesic
$w_n = g_1 \cdots g_n$ RW
$\ell := \lim_{n\to\infty} \frac{d(w_n x, x)}{n}$ a.e.
If μ has finite 1^{st} moment $\left(\iff \int_G d(gx,x) d\mu(g) \right)$
Then $a_n = d(w_n x, x)$ is an integrable, subadditive cocycle, hence by Kingman, $\ell(w_n) = \lim_{n\to\infty} \frac{d(w_n x, x)}{n}$ exists a.s. and is shift-invariant (in the space of increments) .
$T : G^{\mathbb{N}} \to G^{\mathbb{N}}$
$(g_0, \ldots, g_n) \mapsto (g_1, \ldots, g_{n+1}, \ldots)$

$$w_n = g_0 \cdots g_n \widetilde{w_n} = g_1 \cdots g_{n+1}$$

$$\ell(T(w_n)) = \lim_{n\to\infty} \frac{d(g_1 \cdots g_{n+1} x, x)}{n} \overset{?}{=} \lim_{n\to\infty} \frac{d(g_0 \cdots g_n x, x)}{n}$$

$$\frac{d(x, g_0 \cdots g_n x, x)}{n} \le \frac{d(x, g_0 x)}{n} + \frac{d(g_0 x, g_0 g_1 \ldots g_n x)}{n} = \frac{d(x, g_0 x)}{n} + \frac{d(x, g_1 \cdots g_n x)}{n}$$

$$\therefore\ \ell((w_n)) = \lim_{n\to\infty} \frac{d(x, g_0 \cdots g_n x)}{n+1} \le \lim_{n\to\infty} \frac{d(x, g_1 \cdots g_n x)}{n} = \ell(T(w_n))$$

We say $\overline{X} = X \cup \partial X$ is a bordification of X if: $\overline{X}$ is a metrizable space on which G acts by homeo. extending the action of G on X and X is dense in $\overline{X}$.

Theorem 18.1. (T)
Suppose a countable G acts by isometries on (X,d) , and $\overline{X} = X \cup \partial X$ is a bordification, and let $\mu \in P(G)$ of finite 1^{st} moment. Suppose moreover: $x \in X$, and:
1.
- For a.e. (w_n), the limit $\lim_{n\to\infty} w_n x$ exists in ∂X.
- For μ a.e. (w_{-n}), the limit $\lim_{n\to\infty} w_n x$ exists in ∂X
2.
There exists a $\nu \times \check{\nu}$ a.e. defined, G-equivariant map
$P : \partial X \times \partial X \longrightarrow \Gamma X =$ space of biinfinite geodesics .

Definition 18.1 (Backward RW)**.** $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$

$(G^{\mathbb{Z}} = (g_{-1}g_0g_1, \ldots)$
$w_n = g_0 \cdots g_n$
$w_{-n} = g_{-1}^{-1} \cdots g_{-n}^{-1}$
The Back RW for μ is equivalent to (Forward) RW for measure

$$\mu^{\vee}(g) = \mu(g^{-1})$$

Then there exists $\ell \geq 0$ s.t. for a.e. (w_n) there exists a geodesic ray $\gamma : [0, \infty) \to X$ s.t.

$$\frac{d(w_n x, \gamma(\ell_n))}{n} \longrightarrow 0$$

$$\nu(A) = \mathbb{P}\left(\lim_{n\to\infty} w_n x \in A\right)$$
$$\overset{\vee}{\nu}(A) = \mathbb{P}\left(\lim_{n\to\infty} w_{-n} x \in A\right)$$

Corollary 18.1. If $G \to X$ has exponentially bounded growth, then $\partial X, \nu)$ is a model for the Poisson-F boundary.

Proof.
$\mathrm{bnd}^+((g)) = \lim_{n\to\infty} g_0 g_1 \cdots g_n x \in \partial X$
$\mathrm{bnd}^-(g) = \lim_{n\to\infty} g_{-1}^{-1} g_{-2}^{-1} \cdots g_{-n}^{-1} x \in \partial X$
Define map:
$(G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to \partial X \times \partial X \xrightarrow{P} \Gamma X \longrightarrow \mathbb{R}$
$(g) \mapsto \left(\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)\right) \mapsto [\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)] \to d(x, [\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)]\Big)$
$D : (G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to \mathbb{R}$ measurable and a.e. defined .

$\mathrm{bnd}^+(\sigma^n g) = \lim_{k\to\infty} g_n g_{n+1} \cdots g_{n+k} x = w_n^{-1} \mathrm{bnd}^+(g)$
$\mathrm{bnd}^-(\sigma^n g) = w_n^{-1} \mathrm{bnd}^-(g))$
$\therefore$

$$\begin{aligned} D(\sigma^n(g)) &= d(x, [\mathrm{bnd}^+(\sigma^n g), \mathrm{bnd}^-(\sigma^n g)]) \\ &= d(x, w_n^{-1}[\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)]) \\ &= d(w_n x, [\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)]) \end{aligned}$$

□

Goal: a.s. $\lim_{n\to\infty} \frac{d(w_n x, [\mathrm{bnd}^+(g), \mathrm{bnd}^-(g)])}{n} = \lim_{n\to\infty} \frac{D(\sigma^n(g))}{n} = 0$

Lemma 18.1. Let (X, λ) measure space, $T : X \to X$ measure preserving and ergodic.

Let $f : (X, \lambda) \to \mathbb{R}^{\geq 0}$ measurable, and such that

$$g = f \circ T - f \text{ is in } L^1(X, \lambda)$$

.

Then for a.e. $w \in X$,

$$\lim_{n\to\infty} \frac{f(T^n w)}{n} = 0$$

Lemma $\implies$ Theorem
$(X, \lambda) = (G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to \mathbb{R}^{\geq 0}$
$f = D$
$T = \sigma$ (invariant shift)
$f \circ T - f = D \circ \sigma - D$

$$\begin{aligned} |D\sigma(g) - D(g)| &= |d(w_1 x, [\mathrm{bnd}^+, \mathrm{bnd}^-]) - d(x, [\mathrm{bnd}^+, \mathrm{bnd}^-])| \\ &\leq d(w_1 x, x) \end{aligned}$$

is in L^1 by finite 1^{st} moment .

Proof of Lemma.
$g + g\circ T + \cdots + g\circ T^{n-1} = (f\circ T - f) + (f\circ T^2 - f\circ T) + \cdots + (f\circ T^n - f\circ T^{n-1}) = f\circ T^n - f$
$\frac{g + g\circ T + \cdots + g\circ T^{n-1}}{n} = \frac{f\circ T^n - f}{n}$
By Birkhoff,
$\lim_{n\to\infty} \frac{g(\omega) + \cdots + g\circ T^{n-1}(\omega)}{n} = \lim_{n\to\infty} \frac{f(T^n\omega)}{n}$ a.e.
Since f is a.e. finite, $\exists C > 0$ s.t.
$X_C = \{\omega \in X : f(\omega) \leq C\}$ has positive measure : $\lambda(X_c) > 0$
By Poincare recurrence: for a.e. $\omega \in X_c$, there exists infinitely many $n \in \mathbb{N}$ s.t. $T^n(\omega) \in X_C(g)$
$\liminf_{n\to\infty} \frac{f(T^n\omega)}{n} \leq \liminf_{n\to\infty} \frac{C}{n} = 0$
But lim exists, so $\lim_{n\to\infty} \frac{f(T^n\omega)}{n} = 0$ for a.e. $\omega \in X_C = 0$.

$\implies$ by ergodicity, $\lim_{n\to\infty} \frac{f(T^n\omega)}{n} = 0$ a.e. ✓ .

□

19 Ray Approximation and Strip Approximation

$G \to G$, (X, d)
$x \subset \overline{X} = X \cup \partial X$
Assume RW converges to ∂X .
Pick $x \in X$.
For a.e. (ω_n) there is a limit $\lim_{n\to\infty} \omega_n x = \xi \in \partial X$.
$\text{bnd}(g) = \lim_{n\to\infty} \omega_n x$
$\text{bnd} : G^{\mathbb{N}} \to \partial X$

19.1 The Ray Approximation Criterion

Suppose (G, μ) has finite 1^{st} moment and action as exponentially bounded growth.
i.e.

$$\#\{g : gx \in B(x, R)\} \leq C^R$$

for some $C \geq 1$.
Suppose there is a family of measurable maps $\pi_n : \partial X \to G$ such that for $\mathbb{P}$ a.e. (ω_n)

$$\frac{d(\omega_n x, \pi_n(\text{bnd}(g))x)}{n} \to 0$$

Then $(\partial X, \nu)$ is a model for the Poisson boundary of (G, μ) .
$\nu(A) = \mathbb{P}(\lim_{n\to\infty} \omega_n x \in A)$

Proof.
Note: Finite 1^{st} moment & exp. bounded growth $\implies$ finite entropy.
Entropy criterion: $(\partial X, \nu)$ is the PF boundary iff $h(\mathbb{P}^\xi) = 0$ for ν a.e.
Where $\mathbb{P}^\xi$ = conditional probability of $\mathbb{P}$ given that

$$\lim_{n\to\infty} \omega_n x = \xi$$
$$p : (\Omega, \mathbb{P}) \to (\partial X, \nu)$$
$$\mathbb{P} = \int \mathbb{P}^\xi \, d\nu(\xi)$$

□

19.2 Proof Criterion

Let η be the partition of $(\Omega, \mathbb{P})$ which gives the PF boundary:

$$(B_{\text{PF}}, \nu_{\text{PF}}) = \frac{(\Omega, \mathbb{P})}{\eta}$$

and η' the partition such that:

$$\frac{(\Omega, \mathbb{P})}{\eta'} = (\partial X, \nu)$$

By construction: $\eta \neq \eta' \implies H(\alpha|\eta') \leq H(\alpha, \eta)$
α = partition in cylinders of depth 1
$h(\mathbb{P}^{\xi}) > H(\alpha|\eta) - H(\alpha, \eta')$
The conditional random walk is a markov process

$$p^{\xi}(x, y) = \mu(x^{-1}y)\ dy\nu(\xi)$$

$\mu(x^{-1}y)$ is the usual RW from $x \to y$ $dy\nu$ is the RN derivative $x, y \in G$.
By taking the log:

$$\frac{1}{n}\log(\mathbb{P}^{\xi}(C^n_{\omega_n})) = \frac{1}{n}\log(\underbrace{\mathbb{P}(C^n_{\omega_n})}_{\mu^{(n)}(\omega_n)}) + \frac{1}{n}\log\left(\frac{d\omega_n\nu}{d\nu}(\xi)\right)$$

C^n_g = cylinders of depth n containing g .

$$H(\mu^n) = -\sum \mu^n(g)\log\mu^n(g) = \mathbb{E}_{\mu^{(n)}}(-\log\mu^{(n)})$$

We have that $a_n(g) = -\log\mu^{(n)}(\omega_n)$ is a subadditive co-cycle as

$$\mu^{(n+m)}(\omega_{(n+m)}) \geq \mu^{(n)}(\omega_n)\mu^{(m)}(\omega_n)$$

Then by Kingman the limit exists and is:

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}(-\log\mu^{(n)}) = H_{\infty}(\mu)$$

which we do since

$$\lim_{n\to\infty} -\frac{1}{n}\log\mu^{(n)}(\omega_n)$$

is a number so integration will help us find the number.

$$\therefore \lim_{n\to\infty} -\frac{1}{n}\log \mu^{(n)}(\omega_n) = H_\infty(\mu)$$

$$\therefore \frac{1}{n}\log(\mathbb{P}^\xi(C^n_{\omega_n})) = \frac{1}{n}\log(\mathbb{P}(C^n_{\omega_n})) + \frac{1}{n}\log\left(\frac{d\omega_n\nu}{d\nu}(\xi)\right)$$

(Follows from the same proof as (2))

$$-h(\mathbb{P}^\xi) = -H_\infty(\mu) + H(\mu) - H(\alpha|\eta') \tag{1}$$

But

$$H(\alpha|\eta) = H(\mu) - H_\infty(\mu) \tag{2}$$

(1) + (2) $\implies h(\mathbb{P}^\xi) = H(\alpha|\eta) - H(\alpha|\eta')$

Ray Approximation $\implies$ Poisson Boundary

Fix $\epsilon > 0$.

Let A_n be the ball around $\pi_n X$ of radius ϵn .

$\#A_n \le C^{\epsilon n} \implies \frac{\log(\#A_n)}{n} \le \epsilon$

(if supp$\mu \subset A$ then $H(\mu) \le \log \#A$)

$\underbrace{\frac{d(\omega_n x, \pi_n x)}{n}}_{\iff \omega_n \in A_n} \le \epsilon$ for all $n \ge N$ on a set of $\mathbb{P} > 1 - \epsilon'$

$$h(\mathbb{P}^\xi) = \lim_{n\to\infty}\frac{1}{n}H(\mu_n^\xi)$$

$$H(\mu_n^\xi) = -\sum_{g\in A_n}\mu_n^\xi(g)\log\mu_n^\xi(g) - \sum_{g\notin A_n}\mu_n^\xi(g)\log\mu_n^\xi(g)$$

$$\mu(A)\sum_{g\in A_n}\frac{\mu_n(g)}{\mu(A)}\log\left(\frac{1}{\mu(g)}\right) \overset{Jensen}{\le} \mu(A)\log\left(\sum_{g\in A}\frac{\mu(g)}{\mu(A)}\frac{1}{\mu(g)}\right)$$

$$\le \mu(A)\log\left(\frac{\#A}{\mu(A)}\right) \le \log(2\#A)$$

($H(\mu_n^\xi)$ is the conditional distribution of ω_n)

Now for the other one ...

μ has finite first moment.

If μ has finite support:

$b = \inf\{\mu(g) : \mu(g) > 0\} > 0$

$\mu_n(g) \ge \mu(g_1)\mu(g_2)\cdots\mu(g_n) \ge b^n$

$$\therefore -\log \mu_n(g) \leq n \cdot b$$

$$-\sum_{g \in B} \mu_n^\xi(g) \log \mu_n^\xi(g) \leq \mu_n^\xi(B) b_n$$

$$\leq \epsilon' b \cdot n$$

$$\therefore \; H(\mu_n^\xi) = -\sum_{g \in A_n} \mu_n^\xi(g) \log \mu_n^\xi(g) - \sum_{g \in A_n} \mu_n^\xi(g) \log \mu_n^\xi(g)$$

$$\leq \log(2\#A) + \epsilon' \cdot b \cdot n$$

$$\therefore \; \frac{H(\mu_n^\xi)}{n} \leq \frac{\log(2\#A)}{n} + \epsilon' \cdot b$$

$$\therefore \; \limsup \frac{H(\mu_n^\xi)}{n} \leq \epsilon + \epsilon' \cdot b \to 0 \text{ as } \epsilon, \epsilon' \to 0$$

$$\implies h(\mathbb{P}^\xi) = 0 \text{ for a.e. } \xi$$

$$\implies h(\mathbb{P}^\xi) = 0 = H(\alpha|\eta) - H(\alpha|\eta')$$

$$\implies H(\alpha_k|\eta) = H(\alpha_k|\eta') \text{ for all } k \geq 0$$

$$\implies \eta = \eta'$$

$$\implies (\partial X, \nu) \text{ is the Poisson boundary}$$

19.3 The Strip Approximation Criterion

Let μ have finite entropy and finite log moment. Suppose there is a $G-$equivalent map, $\nu \times \nu$ are a.e. defined.

$$S : \partial X \times \partial X \to 2^G$$

s.t. for a.e. $(b_-, b_+) \in \partial X \times \partial X$

$$\frac{1}{n} \log |S(b_-, b_+) \cap B(1, d(\omega_n x, x)| \to 0$$

in probability, then $(\partial X, \nu)$ is the Poisson boundary.
(Assume: $\lim_{n\to\infty} \omega_n x = b_+$ exists a.s. $\lim_{n\to\infty} \omega_n x = b_-$ exists a.e.)
$\check{\nu}$ = hitting measure for backwards random walk.

20 Applications to Random Walks on Hyperbolic Spaces

Theorem 20.1 (Maher, T)**.** Let (X, d) be a geodesic, metric space. X is δ-hyperbolic if geodesic triangles are δ-thin. Let $x_0 \in X$.
Define:

$$(x, y)_{x_0} := \frac{1}{2}\left(d(x_0, x) + d(x_0, y) - d(x, y)\right)$$

(Gromov product)
If X is δ-hyperbolic $\implies (x, y)_{x_0} = d(x_0, [y, z]) + O(\delta)$
$O(\delta)$ is a bdd constant which depends on δ .

20.1 The Gromov boundary ∂X

If X is proper (closed balls are compact)
$\gamma_1 \sim \gamma_2$ if $\sup_{t\geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$.
Example 20.1. $X = \mathbb{R}$ and $\partial X = \{-\infty, \infty\}$.
$X =$ ladder and $\partial X = \{\infty, -\infty\}$.

In general (X not necessarily proper)
Say $(x_n) \subset X$ is a Gromov sequence .
If $\liminf_{m,n\to\infty} (x_n \cdot x_m)_{x_0} = \infty$
Two Gromov sequences $(x_n), (y_n)$ are equivalent if $\liminf_{n\to\infty} (x_n, y_n)_{x_0} = \infty$ In general
$\partial X := \{(x_n) \text{ Gromov sequence }\}$
Theorem 20.2. ∂X is a metric space.
$\eta, \xi \in \partial X$
$\eta = [x_n], \xi = [y_n]$
$(\eta \cdot \xi)_{x_0} = \sup_{x_n\to\eta, y_n\to\xi} \liminf_{m,n} (x_m \cdot y_n)_{x_0}$
Pick $\epsilon > 0 : \rho(\xi, \eta) = e^{-\epsilon(\eta\cdot\xi)_{x_0}}$
This is not yet a metric (no triangle inequality.
$d(\xi, \eta) := \inf \sum_{i=1}^{n-1} \rho(\xi_i, \xi_{i+1})$
where the inf is taken along all finite chains $\xi = \xi_0, \xi_1, \cdots, \xi_{n-1}, \eta = \xi_n$

Lemma 20.1. $\exists C = C(\epsilon)$ s.t.
$Cp(\xi,\eta) \le d(\xi,\eta) \le \rho(\xi,\eta) \quad \forall \xi,\eta \in \partial X$
Setup:
G a countable group of isometries of (X,d) geodesic, metric, δ-hyp. separable space

Example 20.2. $X = \mathbb{R}\checkmark$
$X =$ tree $\checkmark$
$G = \mathbb{F}_2, X = \text{Cay}(\mathbb{F}_2, S)\checkmark$

Definition 20.1. G is *word hyperbolic* if there is a finite set S of generators s.t. Cay(G,S) is S-hyperbolic .

Note: The fact that G is word hyperbolic does not depend on choice of S .
$\underset{\text{countable}}{G} < \text{Isom}(\mathbb{H}^n)$ (where $\mathbb{H}^n$ is an n-dimensional hyperbolic space)
For $X = \mathbb{H}^n$, G need not be word hyperbolic
If the action of G on $\mathbb{H}^n$ is properly discontinuous and cocompact, then G is word hyperbolic.
Properly disc := for any $x \in X, \exists U \ni x$ s.t. $\#\{g \in G : gU \cap U \neq \varnothing\}$ is finite

Example 20.3. $S =$ surface of genus $g \ge 2$.
$\pi_1(s)$ is word hyperbolic.
$\widetilde{S} \simeq \mathbb{D} \simeq \mathbb{H}^2$ there is a regular 8-gon in $\mathbb{H}^2$ with angles $\dfrac{2\pi}{8}$
$\mathbb{H}^2 / G = S$
$G = \langle g_1, g_2, g_3, g_4 \rangle = \pi_1(S)$
The action of G on $\mathbb{H}^2$ is properly discontinuous and cocompact.

Lemma 20.2 (Svarc - Milnor)**.** If G acts properly discontinuously and cocompactly on a δ-hyp. space, then G is word hyperbolic.

Note: if $G < \text{Isom}(\mathbb{H}^3)$ which acts properly but not cocompactly, then G need not be word hyperbolic (it may contain $\mathbb{Z}^2$ $G = \text{Mod}(S)$ mapping class group. Let S closed, orientable, of genus $g \ge 2$.
$\text{Mod}(S) := \text{Homeo}^+(S) / \text{isotopy}$ is a countable, finitely generated group.
(not word hyperbolic)
Dehn twist D_α
Fix curve α on S .
$\langle D_\alpha, D_\beta \rangle = \mathbb{Z}^2$.
It acts on $X = C(S)$ (curve graph)

Vertices = isotopy classes of simple closed curves on S .
Edge if α and β have disjoint representatives

Theorem 20.3. The curve graph is δ-hyperbolic (Masur-Minsky) .
$X =$ free factor complex splitting

If X is proper, then ∂X is compact metric but if X is not proper, then ∂X need not be compact.

Example 20.4. $X = \mathbb{N} \times \mathbb{R}^{\geq 0} \big/ (n,0) \sim (m,0)$
$\partial X \simeq \mathbb{N}$ not compact

20.2 The Horofunction Boundary

Pick $x_0 \in X$ base point.
$z \in X$:
$\rho_z(x) = d(x,z) - d(x_0,z)$
Then $\rho_z(x)$ is 1-Lipschitz and $\rho_z(x_0) = 0$
Consider space $\mathrm{Lip}^1_{x_0}(X) = \{f : X \to \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq d(x,y), f(x_0) = 0\}$ with topology of pointwisde convergence

Proof.
$h \in \mathrm{Lip}^1_{x_0}(X)$
$|h(x)| \leq |h(x) - h(x_0)| \leq d(x,x_0)$
$\mathrm{Lip}^1_{x_0}(X) \subset \otimes_{x \in X} [-d(x,x_0), d(x,x_0)]$
which is compact by Tychonoff

□

Definition 20.2. The horofunction compactification of (X,d) is the closure $\overline{X}^h := \overline{\rho(X)}$ in $\mathrm{Lip}^1_{x_0}(X)$
where $\rho : X \to \mathrm{Lip}^1_{x_0}(X)$
$z \mapsto \rho_z$

21 Random Walks on Weakly Hyperbolic Groups

Let $G \to (X,d)$ geodesic, δ-hyperbolic, separable metric space.
$\mu \in P(G)$ is non-elementary if $\mathrm{sgr}(\mu)$ contain 2 independent hyperbolic (or loxodronic)

Definition 21.1. g is hyperbolic if

$$\tau(g) = \lim_{n\to\infty} \frac{d(g^n x, x)}{n} > 0 \text{ for some } x.$$

In particular, it has 2 fixed points on ∂X, one attracting, one repelling.

Theorem 21.1 (Maher-Tiozzo)**.** In this setup:

1. For a.e. (w_n) and every $x \in X$

$$\lim_{n\to\infty} w_n x = \xi \in \partial X \text{ exists}$$

2. $\exists L > 0$ s.t.

$$\liminf_{n\to\infty} \frac{d(w_n x, x}{n} \geq L > 0$$

If μ has finite 1^{st} moment then

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L > 0 \text{ exists a.s.}$$

Is ∂X (Gromov boundary) a model for the Poisson boundary?

Definition 21.2. The action of G on X is acylindrical if for every $K > 0$ there are numbers N, R s.t. $\forall x, y \in X$: if $d(x,y) \geq R$, then $\#\{g : d(x, gx) \leq K \& d(y, gy) \leq K\} \leq N$

Theorem 21.2. (Maher, T) If in addition μ has finite entropy and finite log moment, and the action is acylindrical, then $(\partial X, \nu)$ is a model for the Poisson ∂.

Example 21.1. • G = Right Angled Artin groups

X = extension graph

acylindricality [Kim-Koberda] .

Γ = finite graph

$G = \langle v \in V(\Gamma) : vw = wv \text{if} (v,w) \in E(\Gamma) \rangle$

Corollary 21.1. • G is word hyperbolic ∂X = boundary of Cayley graph

• $G = \text{Mod}(S)$ ∂X = boundary of curve complex Acylindricality [Sela Bowditch]

[Kaimanovich Masur] $G \curvearrowright \tau(S)$

Poisson boundary $\simeq \partial\tau(S)$

• $G = \text{Out}(\mathbb{F}_N)$ [Horbez]

there are many actions on nonproper hyperbolic spaces, but none known which are acylindrical but they are WPD

Sketch of Proof of 1. Problem: Since ∂X need not be compact, you may not be able to find a stationary measure in $P(\partial X)$
Trick: Consider the horofunction compactification (is always compact, metrizable)
$z \in X \implies \rho_z : X \to \mathbb{R}, \text{fix} x_0 \in X$.
$\rho_z(x) : d(x,z) - d(x_0,z)$
$\rho : X \to \text{Lip}^1_{x_0}(X)$ (is compact in topology of pointwise convergence)
$\overline{X}^h := \overline{\rho(x)}$ in $\text{Lip}^1_{x_0}(X)$

□

Corollary 21.2. $P(\overline{X}^h)$ is compact, so it contains a μ-stationary measure.

Example 21.2. $X = \mathbb{R}$

In general:
$h_3(x) = \lim\limits_{z_h \to \xi} \rho_{z_n}\rho_{z_n}(x) = \lim\limits_{t\to\infty}(d(\gamma(t),x) - t)$ in $X = \mathbb{H}^2$, level sets are horoballs.

Example 21.3. X = infinite tree
$\partial X = \mathbb{Z}$
$X = \mathbb{Z} \times \mathbb{R}^{\geq 0} \big/ (n,0) \sim (m,0)$ $h = \lim_n \rho_{z_n} = \rho_{x_0}$

Idea 2
Local minimum map $\Phi : \overline{X}^h \longrightarrow X \cup \partial X$

Lemma A. There exists $\mathcal{C} = \mathcal{C}(\delta)$ s.t. if $h \in \overline{X}^h$ and x, y are 2 minima of h in X, then $d(x,y) \leq \mathcal{C}$

Lemma B. If $\gamma \subset X$ is a biinfinite geodesic and $h \in \overline{X}^h$, then there exists $p \in gamma$ s.t. h is either 1. $h(x) = h(p) + d(x,p) + O(\delta)$ for $x \in \gamma$
2. $h(x) = h(p) + d^+(x,p) + O(\delta)$ where d^+ is signed distance according to some orientation of γ

$B \implies A$
If x, y are min. of h (diagram needed)
Define $\Phi : \overline{X}^h \longrightarrow X \cup \partial X$
Say h is *finite* if $\inf\limits_{x\in X} h(x) > -\infty$
Say h is *infinite* if $\inf\limits_{x\in X} h(x) = \infty$
Let us call X_F, X_∞
If $h \in X_F$, then $\Phi(h) = \min h \in X$ the map is coarsely defined: formally define

$$\Phi := \{z \in X : h(z) \leq \inf h + 1\}$$

$\text{diam}\Phi(h) \leq \mathcal{C}$
If $h \in X_\infty$, then there is $(x_n) \subset$ s.t. $h(x_n) \to -\infty$.
Then:

Lemma 21.1. x_n converges in the Gramov boundary and $\lim_{n\to\infty} x_n \in \partial X$ does not depend on choice of (x_n) .

$\Phi(h) = \lim_{n\to\infty} x_n \in \partial X$
$h \in \overline{X}^h$ $g \in G$
$(g \cdot h)(x) := h(g^{-1}x) - h(g^{-1}x_0)$
Let $\nu \in P(\overline{X}^h)$ stationary
$\overline{\nu} := \phi_* \nu \in P(\partial X)$

Lemma 21.2. $\nu(X_\infty) = 1$

Note: ϕ is not continuous but $\phi|_{x_0}$ is continuous.
$\rho_{z_n} \to \rho_{x_0}$
$\phi(\rho_{z_n}) = z_n \not\to x_0$

Sketch of proof. $\overline{\nu} = \phi_* \nu$ on $P(\partial X)$
MCT: for a.e. w_n
$(w_n)_* \nu \longrightarrow \nu_w \in P(\overline{X}^h)$
Push by $\overline{\phi}_*$
$(w_n)_* \overline{(}\nu) \longrightarrow \overline{(}\nu)_w \in P(\partial X)$
By δ-hyperbolicity:
If $w_n x \longrightarrow \xi \in \partial X$ then $w_n \overline{\nu} \longrightarrow \delta_\xi$

□

Lemma 21.3. For a.e. (w_n), there exists a limit point of $\rho_{w_n} \in X_\infty$ in $\overline{x}^h$.

Then
The sequence $w_n x$ has at least one limit point ξ in ∂X ...
for each limit point ξ , $w_{n_k} \overline{\nu} \longrightarrow \delta_\xi$ ...
there can be only one limit point, as $\lim_{n\to\infty} w_n \overline{\nu}$ exists
(by the prior 3 results)
Hence: the RW converges a.s. to ∂X

22 Appendix

22.1 Ergodic theorems

In order to talk about asymptotic properties of random walks we need to have tools which assure us of the existence of various averages. Ergodic theorems provide such averages.

The most classical ergodic theorem is the *pointwise ergodic theorem* of Birkhoff.

Definition 22.1. A transformation $T : (X,\mu) \to (X,\mu)$ of a measure space (X,μ) is *measure-preserving* if $\mu(A) = \mu(T^{-1}(A))$ for any measurable set A.

Theorem 22.1 (Birkhoff)**.** Let (X,μ) be a measure space with $\mu(X) = 1$, $f : X \to \mathbb{R}$ be a measurable function, and $T : X \to X$ a measure-preserving transformation. If $f \in L^1(X,\mu)$, then the limit

$$\overline{f}(x) := \lim_{n\to\infty} \frac{f(x) + f(T(x)) + \dots + f(T^n(x))}{n}$$

exists for μ-almost every $x \in X$.

We will derive Birkhoff's theorem from the more general *subadditive ergodic theorem* of Kingman.

A function $a : \mathbb{N} \times X \to \mathbb{R}$ is a *subadditive cocycle* if

$$a(n+m,x) \le a(n,x) + a(m,T^n x) \qquad \text{for any } n,m \in \mathbb{N}, x \in X.$$

The cocycle is *integrable* if for any n, the function $a(n,\cdot)$ belongs to $L^1(X,\mu)$. Assume moreover that

$$\inf \frac{1}{n} \int_X a(n,x)\ d\mu(x) > -\infty.$$

Then the following theorem holds.

Theorem 22.2 (Kingman)**.** Under the previous assumptions, there is an integrable, a.t. T-invariant function $\overline{a}$ such that

$$\lim_{n\to\infty} \frac{1}{n} a(n,x) = \overline{a}(x)$$

for almost every $x \in X$. Moreover, the convergence also takes place in L^1.

We will see the proof of this theorem as presented by Karlsson [**?**].

Proof of Birhkoff's theorem. We now see that Birkhoff's ergodic theorem follows as a corollary. In fact, if we let $a(n,x) \coloneqq \sum_{k=0}^{n-1} f(T^k x)$ then

$$a(n+m,x) = \sum_{k=0}^{n+m-1} f(T^k x) = a(n,x) + a(m, T^n x)$$

is actually an *additive cocycle*, thus it is subadditive. □

22.2 Boundary convergence

Lemma 22.1. Let X be a compact (G,μ) space. Then there exists at least one μ-stationary measure on X.

Lemma 22.2. Let ν be a μ-stationary measure on X. Then for almost every sample path (g_n), there exists the limit

$$\nu_\omega \coloneqq \lim_{n\to\infty} (g_n)_\star \nu.$$

Lemma 22.3. The stationary measure is non-atomic on ∂X.

Lemma 22.4. The stationary measure does not charge the interior of X, i.e.

$$\nu(X) = 0.$$

Proof. Suppose this is not the case. Then by ergodicity of the shift we have $\nu(X) = 0$. Thus, there exists a compact subgroup K such that

$$\nu(K) \geq p > 1/2.$$

Thus, since

$$(g_n)_\star \nu \to \nu_\omega$$

and $\nu = \int \nu_\omega \, d\mathbb{P}$, so there exists a set then for all n large one has

$$g_n^{-1}(K) \geq p,$$

hence

$$K \cap g_n^{-1}(K) \neq \varnothing.$$

□

22.3 Conditional expectation

Theorem 22.3 (Radon-Nikodym)**.** Let $(X, \mathcal{A}, \mu)$ be a probability space, and let ν be a probability measure on $\mathcal{A}$ which is absolutely continuous with respect to μ. Then there exists a function $f \in L^1(X, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu.$$

Let us now consider a probability space $(X, \mathcal{A}, \mu)$, and $\mathcal{B} \subset \mathcal{A}$ a smaller σ-algebra. Then the *conditional expectation* of a function $f \in L^1(X, \mathcal{A}, \mu)$ with respect to $\mathcal{B}$ is a function $g \in L^1(X, \mathcal{B}, \mu)$ (in particular, g is $\mathcal{B}$-measurable) such that

$$\int_B f \, d\mu = \int_B g \, d\mu \qquad \text{for all } B \in \mathcal{B}.$$

Usually one denotes such a g as $\mathbb{E}(f \mid \mathcal{B})$.

Proof. To prove the existence of conditional expectation, one considers the measure ν on $\mathcal{B}$ defined as

$$\nu(B) := \int_B f \, d\mu$$

Then, by the Radon-Nikodym theorem, the measure ν is abs.cont. with respect to μ, hence the Radon-Nikodym derivative $g = \frac{d\nu}{d\mu}$ is a function in $L^1(X, \mathcal{B}, \mu)$ which satisfies

$$\int_B f \, d\mu = \int_B g \, d\mu \qquad \text{for all } B \in \mathcal{B}$$

as claimed. The uniqueness follows from the fact that two functions whose integrals agree on any set of the σ-algebra must agree almost everywhere (check this!). □

Given a set $\mathcal{F}$ of functions, we denote as $\sigma(\mathcal{F})$ the smallest σ-algebra for which all functions are measurable (i.e. the σ-algebra generated by all preimages of measurable sets) and denote

$$\mathbb{E}(f \mid \mathcal{F})$$

the conditional expectation of f with respect to $\sigma(\mathcal{F})$.

This has the intuitive interpretation of the expectation of f *once you know the values of the variables* $\mathcal{F}$. Consider the toin coss $(X_n) : \{0,1\}^{\mathbb{N}} \to \{+1, -1\}$ where each X_n is i.i.d. and is +1 with prob. 1/2, and −1 with prob. 1/2. Then the σ-algebra $\sigma(X_1, \dots, X_n)$ is the set of functions on Ω which *only depend* on the first n coordinates. Note that:

1. If f is independent of $\mathcal{F}$, then $\mathbb{E}(f \mid \mathcal{F}) = \mathbb{E}(f)$.
2. If f is $\mathcal{F}$-measurable, then $\mathbb{E}(f \mid \mathcal{F}) = f$.

Note that in particular if $T : X \to X$ is a measure-preserving system.

22.4 Martingales

Definition 22.2. A sequence $(X_n) : \Omega \to \mathbb{R}$ of measurable functions is a *martingale* if for any n we have

$$\mathbb{E}(X_{n+1} \mid X_1, \dots, X_n) = X_n.$$

A way to think of a martingale is that X_n is the payoff after n steps in a fair (i.e., zero-sum) game. That is, once you know the outcomes of the first n draws, the expected value of the payoff at step X_{n+1} is the previous payoff X_n.

In the example of the toin coss, $Y_n := X_1 + \cdots + X_n$ is a martingale. In fact

$$\mathbb{E}(Y_{n+1} \mid Y_1, \dots, Y_n) = \mathbb{E}(Y_n + X_{n+1} \mid Y_1, \dots, Y_n) = Y_n + \mathbb{E}(X_{n+1}) = Y_n.$$

22.5 Stationary measures

A metric space M is called a G-space if there exists an action of G on M by homeomorphisms, i.e. a homomorphism $\rho : G \to Homeo(M)$.

Lemma 22.5. Let M be a compact, metric G-space, and μ a stationary measure. Then there exists a μ-stationary measure ν on M.

Lemma 22.6. Let ν be a μ-stationary measure on a (G, μ)-space M. Then for any $f \in L^1(M, \nu)$, the sequence

$$X_n := \int_M f \, d(g_n \nu)$$

is a martingale.

22.6 A bit of functional analysis

Let M be a compact metric space. Then $P(M)$ is the space of probability measures on M. We define convergence in the space of measure by saying that (ν_n) converges to ν in the *weak-* topology* if for any continuous $f : M \to \mathbb{R}$, we have

$$\int f \, d\nu_n \to \int f \, d\nu.$$

Theorem 22.4 (Riesz-Markov-Kakutani)**.** The dual to the space $C(M)$ of continuous functions on the compact metric space M is the space of signed Borel measures on M.

Theorem 22.5. The space $P(M)$ is compact with respect to the weak-$\star$ topology.

Proof. It is a closed subspace in the unit ball of the dual space of $C(M)$, in particular

$$P(M) := \{\varphi \in C(M)^\star \ : \ \varphi \geq 0, \varphi(1) = 1\}$$

We say a functional is *positive* if $\varphi(f) \geq 0$ whenever f is a non-negative function. □

Theorem 22.6 (Alaoglu-Banach)**.** Let V be a normed vector space. Then the unit ball in its dual $V^\star$ is compact with respect to the weak-$\star$ topology.

Proof. Recall that if $\varphi \in V^\star$ belongs to the unit ball, then $|\varphi(v)| \leq \|v\|$ for any $v \in V$. Denote as B the unit ball in V, and $B^\star$ the unit ball in the dual, and consider the map $F : B^\star \to [-1,1]^B$ defined as

$$F(\varphi) := (\phi(v))_{v \in B}$$

The map is injective as a functional is determined by its values on the unit ball. Moreover, by Tychonoff's theorem the cube $[-1,1]^B$ is compact as it is a product of compact spaces, and the image $F(B^\star)$ is closed in $[-1,1]^B$, hence it is also compact. □

References

[1] A. Furman, *Random walks on groups and random transformations*, in *Handbook of dynamical systems, Vol. I*

[2] Y. Benoist, J.-F. Quint, *Random walks on reductive groups*

[3] P. Bougerol, J. Lacroix, *Products of random matrices with applications to Schrödinger operators*

[4] W. Woess, *Random walks on infinite graphs and groups*

[5] H. Furstenberg, *Non-commuting random products*

[6] H. Furstenberg, *Poisson boundaries and envelopes of discrete groups*

[7] V. Kaimanovich, A. Vershik, *Random walks on discrete groups: boundary and entropy*

[8] V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*

[9] M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature*

[10] B. Farb, D. Margalit, *A primer on the mapping class group*

[11] Applications

[12] A. Karlsson, G. Margulis, *A Multiplicative Ergodic Theorem and Nonpositively Curved Spaces*

[13] V. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*

[14] J. Maher, G. Tiozzo, *Random walks on weakly hyperbolic groups*