The Distribution of Roots of a Polynomial

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Motivation

How are the roots of a polynomial distributed in $\mathbb{C}$?
Too vague.

Example 1. Consider $f_n = x^n - 1$
Roots are equi-distributed around the unit circle at the points

$$\{e(j/n) : 0 \leq j \leq n-1\}$$

$e(t) := e^{2\pi int}$

In terms of measure write:

$$\mu_{\{f\}} = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$$

for a polynomial with (not necessarily distinct) roots $z_1, \ldots, z_d$
Let $\nu_{\{|z|=1\}}$ be the Haar measure on the unit circle Then we have

Where δ is the Dirac delta measure
This is effectively the *empirical distribution* of the roots
i.e. the *uniform* distribution

$$\lim_{n\to\infty} \mu_{\{x^n-1\}} = \nu_{\{|z|=1\}}$$

In particular *weakly*

Example 2. Consider $f_n = (x-1)^n$
In this case all the roots are at the same point on the unit circle ($x = 1$)
Therefore

$$\lim_{n\to\infty} \mu_{\{(x-1)^n\}} = \delta_1$$

Example 3. Consider $f_n = x^n - 2$
Here again the roots are equi-distributed in angle
As n gets larger, the more points one has, the more uniformly distributed the roots become
But since $2^{1/n} \to 1$ all roots get closer and closer to the unit circle as $n \to \infty$
Therefore

$$\lim_{\{n\to\infty\}} = \nu_{\{|z|=1\}}$$

How do we distinguish between sequences of polynomials for which the limiting measure of the roots is the Haar measure on the unit circle?
One obvious difference between

$$x^n - a \text{ and } (x-1)^n$$

is that in the former case $a^{1/n} \to 1$ as $n \to \infty$ and the latter case the coefficients grow exponentially in n

We want to prove a result along these lines that

If the coefficients of $f(x) \in \mathbb{C}[x]$ are not too large then $\mu_{\{f\}}$ is not far from $\nu_{\{|z|=1\}}$

Measuring the Size of Coefficients

Let $f(x) = a_d \prod_{j=0}^{d} (x - \alpha_j) = \sum_{j=0}^{d} a_j x^j$.

Where $a_d \cdot a_0 \neq 0$.

What is the best measure of the size of the coefficients of a polynomial?

Definition 1.

$$L(f) := \frac{1}{(|a_d| \cdot |a_0|)^{1/2}} \left(\sum_{j=0}^{d} |a_j| \right)$$

Called the *re-normalized 1-norm*
Invented by Erdős and Turán in 1950

First Main Result

Theorem 1. Suppose $f_1, f_2, \ldots$ is a sequence of polynomials in $\mathbb{C}[x]$ where f_d has degree d and $f_d(0) \neq 0$. If $L(f_d) = e^{o(d)}$ as $d \to \infty$ then

$$\lim_{d \to \infty} \mu_{\{f_d(x)\}} = \nu_{\{|z|=1\}}$$

This follows form the following Lemma and Proposition .

Lemma 1. Suppose that $f_1, f_2, \ldots$ is a sequence of polynomials, where f_d has (not necessarily distinct) roots $\alpha_{d,1}, \alpha_{d,2}, \ldots, \alpha_{d,d}$, all non-zero. If $L(f_d) = e^{o(d)}$ as $d \to \infty$ then $|\alpha_{d,j}| = 1 + o(1)$ for $\{1 + o(1)\}d$ values $j, 1 \leq j \leq d$.

Proof. First introduce an alternative measure

Note that we are using the term "measure" in a different context here
These are not measures in the analytical sense of the word

Definition 2.

$$M(f) := |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\}$$

Called the Mahler measure

Relationship Between $M(f)$ and $L(f)$

Jensen's formula gives:

$$M(f) = \exp\left(\int_0^1 \log|f(e(t))|\, dt\right)$$

So that $|M(f) \leq \max_t |f(e(t))|$

Note that:

$$\begin{aligned} |f(e(t))| &\leq \sum_{j=0}^{d} |a_j||e(jt)| \\ &\leq \sum_{j=0}^{d} |a_j| \\ &= L(f)(|a_d||a_0|)^{1/2} \end{aligned}$$

Therefore $M(f) \leq L(f)(|a_d||a_0|)^{1/2}$
This yields:

$$\begin{aligned} \prod_{j=1}^{d} \max\{|\alpha_j|, |1/\alpha_j|\} &= \prod_{j=1}^{d} \max\{1, |\alpha_j|\} \prod_{j=1}^{d} \max\{1, 1/|\alpha_j|\} \\ &= \frac{M_f}{a_d} \cdot \frac{M(f^*)}{|a_0|} \\ &\leq \frac{(L(f)(|a_d||a_0|)^{1/2})}{|a_d||a_0|} = L(f)^2 \end{aligned}$$

□

Jensen: $\log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{|a_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$
Where the $a_k's$ are zeros of f contained in a closed disk D of radius r
f is analytic in this disc
Establishes a connection between the moduli of the zeros of the function f inside the disk D and the average of $\log|f(z)|$ on the boundary circle $|z| = r$

Where $f^* = x^d \cdot f(1/x)$
$\implies L(f^*) = L(f)$
and $M(f^*) = M(f)$

This shows that most of the roots come in towards the unit circle.
Next we want to show that they are uniformly distributed around the circle.
For a given polynomial f write the roots as $\alpha_j = r_j e(\varphi_j)$.
For $0 \leq \alpha < \beta \leq 1$ define:

i.e. write the roots in *polar form*
It is always possible to write a complex number in polar form

Definition 3.

$$N_f(\alpha, \beta) = \#\{j : 1 \leq d \text{ s.t. } \alpha \leq \{\varphi_j\} < \beta\}$$

$N_f(\alpha, \beta)$ counts the number of roots within the arc traversed between the angles α and β
Recall that φ_j represents the angle corresponding to the root α_j

Proposition 1. For any polynomial f of degree $d > 1$ and any $0 \leq \alpha < \beta \leq 1$ we have:

$$|N_f(\alpha, \beta) - (\beta - \alpha)d| \leq 8\sqrt{d \log L(f)}$$

Theorem 1 follows from Proposition 1 + Lemma 1
Proposition 1 can be deduced from the following lemma .

Lemma 2. Fix $\gamma \in [0,1)$. Suppose that $g(x)$ has degree d with all of its roots on the unit circle, and that $N_g(0,\gamma) = [\gamma \cdot d] + 2 \cdot \delta + 1$ Then $\max_t(g(e(t))| \geq \exp(\Delta^2/4(d+1))$

Lemma 2 $\implies$ Proposition 1

Proof.

Given $f(x) = a_d \prod_{j=1}^{d}(x - \alpha_j) = \sum_{j=0}^{d} a_j x^j$ take $g(x) = \prod_{j=1}^{d}(x - e(\varphi_j))$
This way $N_f(0,\gamma) = N_g(0,\gamma)$
Note that:

$$\begin{aligned} \frac{re(\varphi - e(t)|^2}{r} &= r + \frac{1}{r} - 2\cos(2\pi(\varphi - t)) \\ &\geq 2 - 2\cos(2\pi(\varphi - t)) \\ &= |e(\varphi) - e(t)|^2 \end{aligned}$$

This follows from the fact that $(r-1)^2 \geq 2$ and $r > 0$

Multiply this over the roots $r_j e(\varphi_j)$ of f to obtain:

$$|g(e(t))|^2 \leq \frac{|f(e(t))|^2}{a_d^2 \prod_j |r_j|} \leq \frac{L(f)^2 |a_0 a_d|}{|a_0 a_d|} = L(f)^2$$

Since (from above) $|f(e(t))| \leq L(f)|a_0 a_d|^{1/2}$

Therefore $|f((e(t))| \leq L(f)$
Then by Lemma 2:

$$N_f(0,\gamma) \leq [\gamma d] + 1 + 4\sqrt{(d+1)\log L(f)} \leq \gamma d + 8\sqrt{d \log L(f)}$$

Since $L(f) \geq (|a_0| + |a_d|)/|a_0 a_d|^{1/2} \geq 2$

Let $h(x) = \sum_{j=0}^{d} \overline{a}_j x^j$
Then $L(h) = L(f)$ and $N_f(\gamma, 1) = N_h(0, 1-\gamma)$
Then by the above result:

$$\begin{aligned} N_f(0,\gamma) = d - N_f(\gamma,1) = d - N_h(0,1-\gamma) \\ \geq \gamma d - 8\sqrt{d \log L(h)} \\ = \gamma d - 8\sqrt{d \log L(f)} \end{aligned}$$

□

Now let $h(x) = f(e(\alpha)x) = \sum_j b_j x^j$
So that $b_j = e(j\alpha)a_j$ for each j
With $L(h) = L(f)$ and $N_h(0,\gamma) = N_f(\alpha,\beta)$

Where $\gamma = \beta - \alpha$

$\therefore$ — by replacing f with h — we may assume wlog that $\alpha = 0$
The result then follows.

Sketch of the proof of Lemma 2

The idea is to understand the optimal polynomial
i.e. g satisfying the hypothesis for which $\max_t |g(e(t))|$ is minimal
First show that it takes its maximal value at some point between each pair of roots of $g(x)$ inside the arc in question
Next apply the following result (due to Turán):
there cannot be a zero of $g(x)$ at a distance less than $\pi/2d$ from one of these maximal points
This implies that at least 2Δ roots must lie at the endpoints of the interval
Therefore at least one endpoint has a zero with multiplicity $\geq \Delta$
From basic complex analysis a polynomial with a root of high multiplicity must get large

And thus they obtain their lower bound

Second main result

Theorem 1 is a purely analytical result, in that there are no algebraic requirements on f. It is of more interest in arithmetic to have such requirements; for example if we insist that all of the coefficients of f are integers then the roots of f are the union of various complete sets of conjugates of certain algebraic numbers.
In this circumstance Bilu proved a stronger result (using $M(f)$).

Theorem 2. Suppose that $f_1, f_2, \ldots$ is a sequence of polynomials in $\mathbb{Z}[x]$ where f_d has degree d and $f_d(0) \neq 0$. If $M(f_d) = e^{o(d)}$ as $d \to \infty$ then

$$\lim_{d\to\infty} \mu_{\{f_d\}} = \nu_{\{|z|=1\}}$$

To prove this he introduced the notion of *energy* .

Definition 4. Given a compactly supported measure μ on $\mathbb{C}$ we define *energy* by:

$$E(\mu) := -\int\int \log|z-w| d\mu_z d\mu_w$$

If μ is *finitely*supported at $\{\alpha_1, \ldots, \alpha_d\}$ then define:

$$E'(\mu) := \sum_{i\neq j} \mu(\alpha_i) \log|\alpha_i - \alpha_j|$$

and

$$||\mu|| = \left(\sum_i \mu(\alpha_i)^2\right)^{1/2}$$

Note that $E(\mu) \neq E'(\mu)$ since the $i = j$ terms are missed

The following two Lemmas are used to prove Theorem.
They will be mentioned here without proof.

Lemma 3. If $\{\mu_d\}_{d=1,2,\dots}$ have finite support with $||\mu_d|| \to 0$ as $d \to \infty$ and where the μ_d weakly converge to μ then

$$E(\mu) \leq \liminf E'(\mu_d)$$

Weak convergence means convergence of integrals with respect to the given measure of functions that are continuous and bounded

Lemma 4. If $K \subset \mathbb{C}$ is compact then there exists a unique measure $\nu = \nu_K$ for which $E(\nu)$ is minimized over all measure ν whose support is a subset of K . If $K = \{|z| = 1\}$ then $E(\nu_K) = 0$.

ν_K is called the *equilibrium measure* of K

Proof of Theorem 2

Proof. Suppose $f(x) \in \mathbb{Z}[x]$ has distinct roots $\alpha_1, \dots, \alpha_d$ and lead coefficient a_d .
The discriminant of f is a non-zero integer

$$\text{Disc}(f) := a_d^{2d-2} \prod_{i \neq j} (\alpha - i - \alpha_j)$$

Therefore

$$0 \leq \frac{1}{d^2} \log(\text{Disc}(f)) = \frac{2d-2}{d^2} \log|a_d| = \sum_{i \neq j} \frac{1}{d^2} \log|\alpha_i - \alpha_j|$$
$$\leq \frac{2}{d} \log M(f) - E'(\mu_{\{f\}})$$

Then $E'(\mu_{\{f\}}) \leq (2/d) \log M(f)$
As $f_d(x) \in \mathbb{Z}[x]$ we see that

$$\prod_j \max\{1, |\alpha_j|\} \leq M(f_d) = e^{o(d)}$$

and

$$\prod_j \max\{1, 1/|\alpha_j|\} \leq M(f_d^*) = M(f_d) = e^{o(d)}$$

So $\mu_{\{f_d\}}$ is converging to some measure on the unit circle.
By compactness there must be some subsequence of the f_d such that $\mu_{\{f_D\}}$ converges weakly to some limit μ supported on $\{|z| = 1\}$ on that subsequence
But since $||\mu_d|| = 1/\sqrt{d} \to 0$ as $d \to \infty$ we may apply Lemma 3 to deduce that

$$E(\mu) \leq \liminf E'(\mu_d) \leq \liminf (2/d) \log M(f_d) = 0$$

By Lemma 4:

$$E(\mu) \geq E(\nu_{\{|z|=1\}} = 0$$

Therefore $E(\mu) = 0$.
Again by Lemma 4 this implies that $\mu = \nu_{\{|z|=1\}} = 0$

And this is true for any convergent subsequence
By this compactness argument f_d belongs to some convergent subsequence.
The result follows from the fact that they all have the same limiting measure. □

Lower bounds on heights

Kronecker established a result on roots of unity on the unit circle which can be reinterpreted as stating that for any integer $d \geq 1$ if $M(\alpha) > 1$ then there exists a constant $\delta(d) > 0$ such that $M(\alpha) \geq 1 + \delta(d)$ for all α of degree d
In 1933 Lehmer made the extraordinary conjecture that $\delta(d) \geq \delta(10) = 0.1762808\ldots$ obtained from the example where α is a root of

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

In 1979 Dobrowolski showed that one can take:

$$(1 - \epsilon)(\log \log d / \log d)^3$$

but this has not been much improved subsequently.

Compact Sets with Minimal Energy

Let K be a compact subset of the complex plane and suppose $E(\nu_K) = 0$.
Rumely showed that Theorem 2 (Bilu's result) can be extended when appropriately reformulated to *any such* set K

e.g. K = unit circle but there are other interesting examples e.g. the line segment $[-2, 2]$

Rumely prefers to work with the *capacity* of K given by $\exp(-E(\nu_K))$ $= 1$ in this case

Motivating capacity results

If $\alpha_1, \ldots, \alpha_d \in K$ is the support of measure $\mu = \mu_\alpha$ where $\mu_\alpha(\alpha_j) = 1/d$ then

$$E(\nu_k) = \lim_{d \to \infty} \min_{\alpha_a, \ldots, \alpha_d \in K} E'(\mu_\alpha)$$

Called the *transfinite diameter* of a compact set K

One consequence of this is that ν_K is supported only on the outer boundary of K.
A second consequence of this is that

$$E(\nu_K) = \lim_{d \to \infty} \text{monic} \min_{\substack{f(z) \in mathbbC[x] \\ \deg f = d}} \frac{1}{d} \sup_{z \in K} \log |f(z)|$$

The RHS is the logarithm of the Chebyshev constant of K

A "monic" polynomial is a polynomial with leading coefficient equal to 1

Alternative Mahler measure

The Mahler measure